

Research interests—2008  
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There are three underlying themes in my research: (1) I like to study problems using geometric and topological techniques, (2) I strongly prefer concrete problems and explicit examples over abstract theory, and (3) I really enjoy proving statements that are motivated by experimental computer calculation, real-world physics, or both.

Before starting, let me mention that I have a wide variety of interests and that I am often open to new collaborations and new problems. In fact, I have both a French doctorate in hyperbolic geometry and an American Ph.D. in complex dynamics. I also have publications in a third area: applications of dynamical systems theory to plasma physics.

While a significant portion of my previous research was in hyperbolic geometry (see, for example, the list of publications on my CV), most of my focus for the future is in the direction of complex dynamics. Below I will give a very brief background on complex dynamics and then describe two main areas for my future research.

**1.1. A brief tour of Complex Dynamics in several variables.** Major advances have been made during the past 30 years studying the dynamics of holomorphic mappings of the complex numbers  $\mathbb{C}$  or the Riemann Sphere  $\hat{\mathbb{C}}$  using Montel's Theorem, contraction properties of the hyperbolic metric, quasi-conformal mappings, Teichmüller theory, and especially the measurable Riemann Mapping theorem.

Given a rational mapping  $R$  of the  $k$ -dimensional complex projective space  $\mathbb{C}\mathbb{P}^k$ , the fundamental definitions remain the same: a point  $p$  is in the Fatou set of  $R$  if there is a neighborhood  $U$  of  $p$  so that the iterates  $\{R, R^{\circ 2}, \dots\}$  form a normal family on  $U$ . The Julia set  $J(R)$  is the complement of the Fatou set.

However, the difference between dynamics in one complex variable and in two or more complex variables is dramatic. Some of the tools mentioned above have generalizations but only in much more restricted conditions. In hand with this decrease in the number of tools available, new phenomena arise, including indeterminate points [17], the possibility that the degree an iterate  $\deg(R^{\circ n})$  may not equal  $(\deg(R))^n$  [13, 8], and Fatou sets with complicated topology [9, 22, 16]. A nice survey of complex dynamics in many variables is presented in [24].

I propose projects in two areas from dynamics: first to study a class of rational mapping of  $\mathbb{C}\mathbb{P}^2$  arising from statistical physics (Subsection 1.2, below). These mappings provide interesting insight into physics, while pushing the contemporary theory of dynamics in many dimensions.

Second, I propose to develop further applications of the theory of linking closed loops with a closed current, as developed in [16, 22] (Subsection 1.3, below). I think that this type of linking number could become a fundamental new tool with applications to complex dynamics in many variables, and possibly to even more general questions in complex geometry.

**1.2. Renormalization dynamics related to the Ising model on hierarchical lattices.** The renormalization theory from physics has been extremely successful in describing changes of phase for large systems of interacting particles. One typically describes the physical properties for a system of particles on a given physical lattice (finite graph) and then relates them to the physical properties on lattice of finer scale by means of a “Renormalization operator”. Understanding the limit under iteration of this renormalization operator is believed to describe the real microscopic physics, hence there is a natural connection with dynamical systems.

One arena for renormalization theory is magnetic materials and the simplest model in this arena is the Ising model. In the Ising model, at each finite scale the material is described by a lattice to which vertices are assigned spins up (+1) and down (−1) with edges corresponding to interactions between neighboring electrons. For the Ising model, the typical physical properties that are considered are free energy, magnetization, and spontaneous magnetization.

**My previous work (joint with Pavel Bleher and Misha Lyubich):**

We are currently finishing an exciting project [6] whose goal was to study the dynamics of a rational mapping  $\mathcal{R} : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$  given in affine coordinates by

$$(1.1) \quad \mathcal{R}(z, t) = \left( \frac{z^2 + t^2}{z^{-2} + t^2}, \frac{z^2 + z^{-2} + 2}{z^2 + z^{-2} + t^2 + t^{-2}} \right).$$

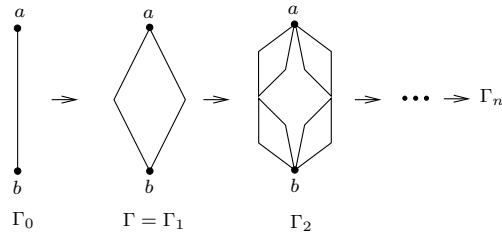


FIGURE 1.1. The Diamond Hierarchical Model

It describes the renormalization operator on for the hierarchy of lattices known as the *Diamond Hierarchical Model*. The first three graphs in the hierarchy are shown in Figure 1.1.

This mapping was studied previously by Bleher and Žalys [5] and Bleher and Lyubich [4]. Here,  $t$  corresponds to temperature (normalized so that physical values correspond to  $t \in [0, 1]$ ) and  $z$  describes an external magnetic field applied to the system. However we generally consider both  $t$  and  $z$  as complex numbers.

Many of the difficult properties for rational mappings that were mentioned in Subsection 1.1 are present for  $\mathcal{R} : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$ , including indeterminate points at  $(1, \pm i)$  (among others) and a drop in degree under iteration, e.g.  $\deg(\mathcal{R}^2) = 28 < 36 = (\deg\mathcal{R})^2$ .

Let me now explain more precisely how  $\mathcal{R}$  is related to physics:

In statistical physics, a function called the partition function encodes many of the physical properties of the system (at thermodynamic equilibrium) and zeros of this function correspond (in an appropriate way) to changes of phase. The renormalization operator relates the zeros of the partition function for the system at one scale with those at the next finer scale. For the Diamond Hierarchical Model we can use the self-similar construction of  $\Gamma_{n+1}$  from  $\Gamma_n$  to write the renormalization operator  $\mathcal{R}$  explicitly, obtaining (1.1).

For (ferromagnetic) Ising models the famous Lee-Yang Theorem [21] states that when  $t \in [0, 1]$  all of the zeros for the partition function lie on the unit circle  $|z| = 1$ . Henceforth, we will refer to the zeros of the partition function that lie on

$$(1.2) \quad \mathcal{C} = \{(z, t) : |z| = 1, t \in [0, 1]\}$$

as the *Lee-Yang zeros*. They form a type of preferred (proper) subset among all of the zeros of the partition function (viewed as a subset of  $\mathbb{C}^2$ ).

The Lee-Yang zeros have special physical significance because the many macroscopic physical properties of the system (e.g. free energy and magnetization) depend only on their limiting distribution, rather than the global distribution in  $\mathbb{C}^2$ .

For the Diamond Hierarchical Model, one can check that  $\mathcal{R}(\mathcal{C}) = \mathcal{C}$  so that the Lee-Yang zeros corresponding to  $\Gamma_{n+1}$  are obtained by pulling back the Lee-Yang zeros corresponding to  $\Gamma_n$  under  $\mathcal{R}$ . Therefore, the limiting distribution of Lee-Yang zeros reduces to studying the dynamics of  $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{C}$ .

This physical theory motivates our current project—we are have used techniques from complex dynamics and complex geometry to prove:

**Theorem 1.1.** (Bleher, Lyubich, R): *The mapping  $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{C}$  is partially hyperbolic.*

Let me explain what the term partially hyperbolic means: Based at each point  $x \in \mathcal{C}$  is a horizontal tangent cone  $\mathcal{K}^h(x) \subset T_x\mathcal{C}$  and a vertical line  $L(x) \subset T_x\mathcal{C}$  depending continuously on  $x$  with the following invariance properties:  $D\mathcal{R}(\mathcal{K}^h(x)) \subset \mathcal{K}^h(\mathcal{R}(x))$  and  $D\mathcal{R}(L(x)) = L(\mathcal{R}(x))$ . Horizontal tangent vectors  $v \in \mathcal{K}^h(x)$  get exponentially stretched under iterates of  $\mathcal{R}$  at a rate that dominates any occasional expansion of tangent vectors in  $L(x)$ .

Note that partially hyperbolic systems is a very popular field of study for current researchers in real dynamics (see, for example [14]). However, usually researchers begin with a system that is partially hyperbolic (either by hypothesis, or for some relatively obvious reason) and study its dynamical properties.

*It is very rare to take a given example (in terms of explicit formulae), such as  $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{C}$ , and prove directly that it is partially hyperbolic.*

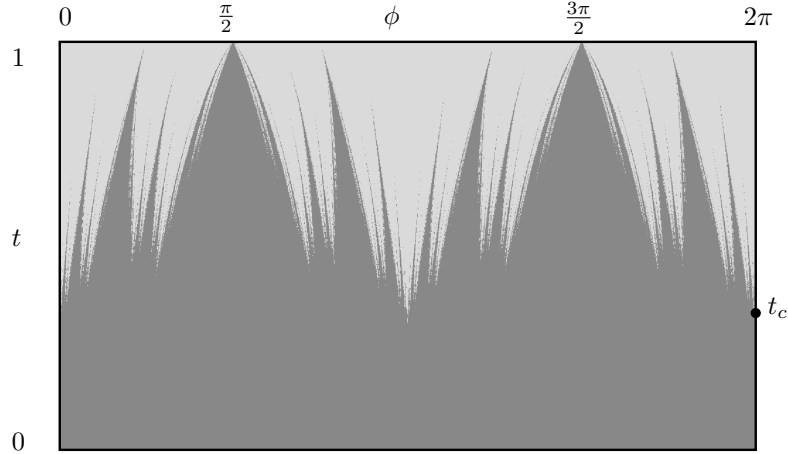


FIGURE 1.2. Dynamics on the invariant cylinder  $\mathcal{C}$ . Points converging to the top are colored light grey and points converging to the bottom are colored dark grey.

One consequence of partial hyperbolicity for  $\mathcal{R}$  is the existence of an  $\mathcal{R}$  invariant vertical foliation  $\mathcal{F}$  on the cylinder  $\mathcal{C}$  called the *central foliation*. Another consequence is that almost every point on  $\mathcal{C}$  has orbit forward asymptotic to either the bottom ( $t = 0$ ) or to the top ( $t = 1$ ) of the cylinder  $\mathcal{C}$ . See Figure 1.2. This is an example of the famous “intertwined basins” phenomenon studied by Kan, Yorke et al [20, 1], and more recently by Bonifant and Milnor [7].

Physically, our work describes the limiting distribution of Lee-Yang zeros related to  $\Gamma_n$  as  $n$  tends to infinity in a dynamical way:

**Theorem 1.2.** (Bleher, Lyubich, R): *The asymptotic distribution of Lee-Yang zeros at a temperature  $t_0 \in [0, 1]$  is obtained by translating the Lebesgue measure from  $t = 0$  up to  $t = t_0$  using holonomy along the leaves of  $\mathcal{F}$ .*

Let me clarify what we mean: through each point  $(\phi, 0)$  on the bottom of  $\mathcal{C}$  is a unique “vertical” leaf  $\mathcal{F}_\phi$  of the foliation  $\mathcal{F}$ . This leaf will intersect the unit circle at height  $t = t_0$  in a unique point  $h(\phi)$ . If we denote the unit circle by  $\mathbb{T}$ , then the resulting mapping  $h : \mathbb{T} \times \{0\} \rightarrow \mathbb{T} \times \{t_0\}$  is called the holonomy map of  $\mathcal{F}$ . The statement of the theorem is that if  $m$  is the Lebesgue measure on  $\mathbb{T} \times \{0\}$  then the Lee-Yang zeros are distributed on  $\mathbb{T} \times \{t_0\}$  according to the push forward measure  $h_*(m)$ .

*We believe that our results are the first rigorous description of the Lee-Yang zeros in terms of both the parameters temperature  $t$  and magnetic field  $z$  for any (non-trivial) model.*

**Proposed future projects:** There are many other forms of hierarchical lattices for which one can write down a renormalization mapping similar to  $\mathcal{R}$ . A detailed description of many possibilities is given in a recent preprint [25] by De Simoi and Marmi.

**Question 1:** *Do similar results to Theorem 1.1 and Theorem 1.2 hold for the Ising model on any hierarchical lattice? Is there an underlying physical mechanism for partial hyperbolicity of any such renormalization mapping?*

Question 1 is interesting from a physical perspective because it may provide more detailed predictions of what to expect for the more realistic  $\mathbb{Z}^d$  lattices. This could possibly even lead to an experimentally verifiable prediction (see, for example [3]).

Studying Question 1 would also be a way to generate specific mappings which will test the current theory of complex dynamics in  $\mathbb{C}\mathbb{P}^2$  and also real-dynamics in two variables (via the Lee-Yang cylinder  $\mathcal{C}$ ). In particular, it may provide many new examples of partially hyperbolic dynamical systems, enriching an active field of contemporary real dynamics.

Our work in [6] has already resulted in many fundamental dynamical questions which test the very limits of the current theory. Let me describe two of these questions:

Suppose that a rational map  $R$  has a hyperbolic invariant circle  $S$ . For example, this occurs if  $R$  has a transversally attracting invariant line  $L$  within which  $R$  is given by  $z \mapsto z^d$  (for some  $d > 1$ ). One would often like to construct a stable manifold  $\mathcal{W}^s(S)$  (consisting of points whose orbits under  $R$  converge to  $S$ ) with  $\mathcal{W}^s(S)$  having nice regularity. However, in this situation  $S$  does not satisfy the stronger hypothesis of *normal hyperbolicity* that is supposed by the classical hyperbolic theory [HPS].

This precise situation occurs in both [6] and [22] in which we have been able to directly prove that  $\mathcal{W}^s(S)$  is a real-analytic manifold. In the former, real analyticity of  $\mathcal{W}^s(S)$  has an important physical interpretation, in terms of the Lee-Yang zeros.

**Question 2:** *Under what conditions does a hyperbolic invariant circle  $S$  for a rational map  $R$  have a real-analytic stable manifold  $\mathcal{W}^s(S)$ ?*

Similar questions can be asked for more general hyperbolic invariant manifolds, as well.

Given a rational map  $R : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$  and an algebraic hypersurface  $C \subset \mathbb{C}\mathbb{P}^k$ , it is often asked whether an appropriately weighted sequence of pullbacks  $(R^n)^*C$  converges to a canonical invariant current (known as the Green's current), independent of the choice of initial  $C$ . (See the appendix from [24] for details on currents and [13, 24] for examples of the types of theorems proven.) In some sense this is a study of the "ergodic properties" of  $R$ . Typical results state that this is the case for a generic choice of the curve  $C$ .

In our work [6], we are interested this same question applied to a *very specific choice of  $C$* , i.e. the zero locus of the partition function for  $\Gamma_0$ . The question is essential to understanding the distribution for zeros of the partition function, globally in  $\mathbb{C}\mathbb{P}^2$ . The general theory is not quite equipped to tell us whether appropriately weighted pullbacks of this specific  $C$  will converge to the Green's current (i.e. whether  $C$  is generic). Instead we have given a direct proof.

**Question 3:** *Is there a convenient characterization (depending on  $C$  and  $R$ ) that a specific algebraic hypersurface  $C$  have (appropriately weighted) sequence of pull-backs under  $(R^n)^*$  converging to the Green's Current?*

Because of the existing work in [6], we are in a nice position to approach Question 3.

**1.3. Linking with closed currents: applications to complex dynamics.** Very little is known about the global topology for Fatou sets  $U(f)$  (see Subsection 1.1) for rational mappings  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$ . More is known about parameter spaces for families of holomorphic maps of a single variable  $f_a(z) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  depending holomorphically on at least two complex parameters  $a \in \mathbb{C}^2$ , but there are still many interesting questions about such parameter spaces.

In this Subsection I'll describe ways to study these objects using the technique of linking a closed curve with a closed current, which was developed by Suzanne Hruska and myself in [16, 22].

**My previous work (joint with Suzanne Hruska):**

In my thesis at Cornell, I developed the notion of *linking number* between a closed loop and a closed (1,1) current, in order to show that the basins of attraction for the roots of a specific Newton's Method have have infinitely generated first homology [22]. The definition is a natural generalization of the classical linking number from geometric topology [23] and it is essentially based on the intersection theory of closed currents [10, 15].

More recently, in joint work with Suzanne Hruska [16] we have adapted ideas from [22] to work more generally for any globally holomorphic mapping  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$ . Such maps are much simpler than the Newton maps considered in [22], but they can also have Fatou sets illustrating remarkable topology.

In particular, we have proven the two following theorems:

Consider a holomorphic mapping  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$  that can be written in affine coordinates as  $f(z, w) = (p(z, w), q(z, w))$  with  $p$  and  $q$  polynomials. Such a mapping is called a *polynomial endomorphism* of  $\mathbb{C}\mathbb{P}^2$ . Note that the line at infinity  $\Pi$  is invariant under a polynomial endomorphism and within this line there is a Julia set  $J_\Pi$  corresponding to the restriction  $f|_\Pi : \Pi \rightarrow \Pi$ .

**Theorem 1.3.** (Hruska, R): *Suppose that  $f$  is a polynomial endomorphism of  $\mathbb{C}\mathbb{P}^2$  with restriction  $f|_\Pi$  to the line at infinity  $\Pi$ . If  $f|_\Pi$  is hyperbolic and  $J_\Pi$  is disconnected, then the Fatou set  $U(f)$  has infinitely generated first homology.*

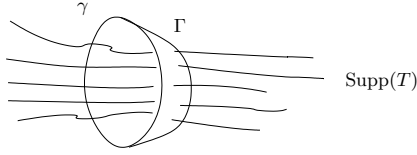


FIGURE 1.3. Cartoon figure of linking  $\gamma$  with  $T$ .

Even simpler are the *polynomial skew products* which occur when  $p(z, w)$  does not depend on  $w$ , that is  $p(z, w) = p(z)$ , see [19, 12]. Associated to  $p(z)$  is a Julia set  $J_p$  within the  $z$ -plane. Within each vertical line  $\{z\} \times \mathbb{C}\mathbb{P}^2$  there is also an appropriate notion of “vertical Julia set”  $J_z$ .

**Theorem 1.4.** (Hruska, R): *Suppose  $f$  is a polynomial skew product with  $J_z$  disconnected for some  $z \in J_p$ , then the Fatou set has infinitely generated first homology.*

In [16] we provide numerous examples of holomorphic maps of  $\mathbb{C}\mathbb{P}^2$  for which either Theorem 1.3 or Theorem 1.4 gives that the Fatou set  $U(f)$  has infinitely generated first homology.

*To my knowledge we provide the first examples of globally holomorphic endomorphisms  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$  for which it is known that the Fatou set has infinitely generated first homology. Furthermore, it is a very common property.*

Let me now explain why the notion of linking a closed loop with a closed current is central to the proofs of Theorems 1.3 and 1.4. If the reader is unfamiliar with currents, he or she may think of them as a measure-theoretic generalization of differential forms, defined in a way analogous to how distributions are defined to generalize functions. (See the appendix from [24] for more background.)

Given any globally holomorphic map  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$ , the standard theory [18, 26, 24] describes the Julia set  $J(f)$  as the support of a  $(1, 1)$  current known as the *Green’s current*  $T$ . Consequently the Fatou set  $U(f)$  is given by  $\mathbb{C}\mathbb{P}^2 \setminus \text{supp}(T)$ .

We can define a pairing between piecewise smooth 2-chains  $\sigma$  and  $T$  by  $\langle \sigma, T \rangle = \int_{\sigma} \eta_T$ , where  $\eta_T$  is a smooth approximation of  $T$  within its cohomology class, see [15, pages 382-385]. This pairing is well-defined and depends only on the cohomology class of  $T$ .

If  $\gamma$  is a smooth curve disjoint from the support of  $T$ , let  $\Gamma$  be a piecewise smooth 2-chain with  $\partial\Gamma = \gamma$ . One could try to define the linking number as:  $lk(\gamma, T) = \langle \Gamma, T \rangle$ . See Figure 1.3.

Notice that we could have chosen any other  $\Gamma'$  with  $\partial\Gamma' = \gamma$ . This leads to the possibility for the linking number not to be well-defined. One can check that within  $\mathbb{C}\mathbb{P}^2$  this definition is only well-defined modulo 1:

**Definition 1.1.** If  $\gamma$  is a piecewise smooth closed curve disjoint from the support of  $T$ , we define the *linking number*  $lk(\gamma, T)$  by

$$lk(\gamma, T) := \langle \Gamma, T \rangle \pmod{1},$$

where  $\Gamma$  is any piecewise smooth two chain with  $\partial\Gamma = \gamma$ .

Unlike the linking numbers between closed loops in  $\mathbb{S}^3$  that are classically studied in topology, it is often the case that  $\langle \Gamma, T \rangle \notin \mathbb{Z}$ , resulting in non-zero linking numbers (mod 1). See [16, Sec. 3] for details.

**Proposed future projects:** Theorems 1.3 and 1.4 are very specific to polynomial endomorphisms and polynomial skew products (respectively). It remains a mystery what is a useful sufficient condition to guarantee that a general holomorphic map  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$  have Fatou set with complicated homology.

For any globally holomorphic map  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$  there is a canonical invariant measure  $\mu_f = T \wedge T$  that is dynamically significant because it maximizes “entropy”. This may be one key to understanding the homology of  $U(f)$ :

**Question 4:** Given a holomorphic mapping  $f : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$ , does  $\text{supp}(\mu_f)$  disconnected imply that  $H_1(U(f))$  is infinitely generated?

We have done an extensive search of the (few) examples for which the structure of  $T$  and  $\mu$  are well understood. However, we have not found any examples for which  $\text{supp}(\mu_f)$  is disconnected and  $U(f)$  has

simple homology. Note that there are also many examples of mappings with  $\text{supp}(\mu_f)$  connected for which  $H_1(U(f))$  is infinitely generated, so an answer to Question 4 would therefore only lead to one interesting sufficient condition, among many possible others.

More recently, closed positive currents have appeared in a description of parameter spaces for holomorphic mappings of one variable. For example, the parameter locus for iteration of cubic polynomials is naturally two-complex dimensional and within this space the bifurcation locus (analogous to the boundary of the Mandelbrot set) has been described as the support of a closed positive  $(1, 1)$  current [11, 2].

**Question 5:** Can the technique of linking with currents be used to find interesting loops within the parameter space of cubic polynomials?

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