

TOPOLOGY OF FATOU COMPONENTS FOR ENDOMORPHISMS OF \mathbb{CP}^2 : LINKING WITH THE GREEN'S CURRENT

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ABSTRACT. Little is known about the global topology of Fatou components for holomorphic endomorphisms $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$. We develop a type of linking number between closed loops in the Fatou set of f with the Green's current T , which forms the complement of the Fatou set. Using these linking numbers we establish that many classes of endomorphisms have Fatou components with infinitely generated first homology.

For example, we prove that the Fatou set has infinitely generated first homology for any polynomial endomorphism of \mathbb{CP}^2 for which the restriction to the line at infinity is hyperbolic and has disconnected Julia set. In addition we show a polynomial skew product of \mathbb{CP}^2 has Fatou set with infinitely generated first homology if the vertical Julia set in an appropriate slice is disconnected. We then conclude with a section of concrete examples and questions for further study.

1. INTRODUCTION

Our primary interest in this paper is the topology of the Fatou set for holomorphic endomorphisms of \mathbb{CP}^2 (written as \mathbb{P}^2 in the remainder of the paper). We show that in many cases the homology, and hence fundamental group, of the Fatou set is infinitely generated. One motivation is to find a generalization of the fundamental dichotomy for polynomial (or rational) maps of the Riemann sphere: the Julia set is either connected, or has infinitely many connected components. Further, this type of result paves the way to an exploration of a potentially rich algebraic structure to the dynamics on the Fatou set.

Given a holomorphic endomorphism $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$, the *Fatou set* $U(f)$ is the maximal open set on which the iterates $\{f^n\}$ form a normal family. The *Julia set* $J(f)$ is the complement, $J(f) = \mathbb{P}^2 \setminus U(f)$. The standard theory [16, 27, 26] gives a convenient description of these sets in terms the *Green's current* T . Specifically, T is a dynamically defined closed positive $(1, 1)$ current with the property that $J(f) = \text{supp}(T)$. We provide relevant background about the Green's current in Section 2. Throughout this paper we assume the degree of f is at least two (i.e. that the components of a lift of f to \mathbb{C}^3 have degree at least two).

Since the Green's current is closed, it carries a cohomology class $[T] \in H^2(\mathbb{P}^2)$. In Section 3, we develop a theory of linking number between a closed loop in \mathbb{P}^2 and any closed positive $(1, 1)$ current. The techniques are based on a somewhat similar theory in [24], developed by the second author of the present article. We then apply this theory to closed loops in $U(f) = \mathbb{P}^2 \setminus \text{supp } T$, and show a non-zero linking number, denoted $lk(\gamma, T)$, provides a sufficient condition for a loop $\gamma \subset U(f)$ to represent a non-trivial homology class.

In many cases pulling back a non-trivial loop γ under iterates of f allows one to prove that $H_1(U(f))$ is actually infinitely generated. In Section 4 we consider this possibility in the special case that γ lies in the basin of attraction for an attracting periodic point.

Date: October 9, 2008.

¹Research supported in part by a grant from the University of Wisconsin Milwaukee.

Combining these results, we obtain

Theorem 1.1. *Suppose that f is a holomorphic endomorphism of \mathbb{P}^2 and that γ is a smooth loop in the Fatou set $U(f)$ with $lk(\gamma, T) \neq 0$. Then $[\gamma] \neq 0 \in H_1(U(f))$.*

Suppose further that γ is contained in the basin of attraction for an attracting cycle of f and consider a sequence of lifts $\gamma_0 = \gamma, \gamma_1, \gamma_2, \dots$ with $f_([\gamma_i]) = [\gamma_{i-1}]$ for each i . If $lk(\gamma_i, T) \neq 0$ for each i , then $H_1(U(f))$ is infinitely generated.*

An immediate consequence is that under the same hypotheses the Fatou set $U(f)$ has infinitely generated fundamental group. (This follows from the Hurewicz Theorem, see [13, 6]).

In order to apply this theory, one needs a detailed knowledge of the geometry of T . In the second half of the paper we consider two situations where Theorem 1.1 can be readily applied to provide examples of endomorphism f of \mathbb{P}^2 which have Fatou set $U(f)$ with infinitely generated homology.

The first situation is for polynomial endomorphisms of \mathbb{P}^2 , that is, holomorphic maps of \mathbb{P}^2 that are obtained as the extension a polynomial map $f(z, w) = (p(z, w), q(z, w))$ on \mathbb{C}^2 . For these mappings, the line at infinity, denoted by Π , is totally invariant and superattracting. Therefore the restriction of T to Π is easy to understand using the dynamics of the resulting rational map of $f|_{\Pi}$ and its Julia set J_{Π} . In Section 5 we prove the following theorem.

Theorem 1.2. *Suppose that f is a polynomial endomorphism of \mathbb{P}^2 with restriction $f|_{\Pi}$ to the line at infinity Π . If $f|_{\Pi}$ is hyperbolic and J_{Π} is disconnected, then the Fatou set $U(f)$ has infinitely generated first homology.*

This theorem provides for many examples of polynomial endomorphisms f of \mathbb{P}^2 with interesting homology of $U(f)$. We present one concrete family to which Theorem 1.2 applies in Example 5.3.

We then consider the special family of polynomial endomorphisms known as polynomial skew products. While Theorem 1.2 applies to certain polynomial skew products, we develop additional sufficient criteria for $U(f)$ to have interesting homology.

A polynomial skew product is a map of \mathbb{C}^2 having the form $f(z, w) = (p(z), q(z, w))$, where p and q are polynomials. We assume that $\deg(p) = \deg(q) = d$ and $p(z) = z^d + O(z^{d-1})$ and $q(z) = w^d + O_z(w^{d-1})$, where we have normalized leading coefficients. Since f preserves the family of vertical lines $\{z\} \times \mathbb{C}$, one can analyze f via the collection of one variable fiber maps $q_z(w) = q(z, w)$, for each $z \in \mathbb{C}$. For this reason polynomial skew products provide an accessible generalization of one variable dynamics to two variables and have been previously studied by many authors, including Jonsson in [19] and DeMarco, together with the first author of this paper, in [8].

For any polynomial map which extends holomorphically to \mathbb{P}^2 the Green's current can be computed in the affine coordinates on \mathbb{C}^2 as: $T := \frac{1}{2\pi} dd^c G_{\text{affine}}$, where G_{affine} is the (affine) Green's function defined by $G_{\text{affine}}(z, w) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log_+ \|f^n(z, w)\|$. The "base map" $p(z)$ has a Julia set $J_p \subset \mathbb{C}$ and, similarly, a Green's function $G_p(z) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log_+ \|p^n(z)\|$. Furthermore, one can define a fiber-wise Green's function by: $G_z(w) := G_{\text{affine}}(z, w) - G_p(z)$,¹ allowing for definitions of the fiber-wise Julia sets: $K_z := \{G_z(w) = 0\}$ and

¹For the purist: the Green's functions G_p and G_z should also have the subscript "affine", but it is dropped here for ease of notation. See Section 2 for the distinction.

$J_z := \partial K_z$. More details on Green's functions and the Green's current are given in Section 2.

In Section 6, starting with a little background on polynomial skew products, we establish a lemma which gives us some control over the geometry of $\text{supp } T$, then apply this knowledge to prove:

Theorem 1.3. *Suppose f is a polynomial skew product and that for some $z \in J_p$, J_z is disconnected. Then the Fatou set has infinitely generated first homology.*

For any endomorphism there is also the measure of maximal entropy $\mu = T \wedge T$. Thus another candidate for the name ‘‘Julia set’’ is $J_2 := \text{supp}(\mu)$. The Julia set that is defined as the complement of the Fatou set is sometimes denoted by J_1 , to distinguish it from J_2 .

The condition from Theorem 1.3 that for some $z \in J_p$, J_z is disconnected might seem somewhat unnatural. A seemingly more natural condition might be that J_2 is disconnected, since for polynomial skew products it is known (see [19]) that $J_2 = \overline{\bigcup_{z \in J_p} J_z}$. However, in Example 7.1 we present certain polynomial skew products with J_2 connected, but with the Fatou set having infinitely generated first homology. (These examples are obtained by applying Theorem 1.3 to examples from [19] and [8].) In fact, some of these examples persist over an open set within a one-variable holomorphic family of polynomial skew products. Therefore, for polynomial skew products, connectivity of the fiber Julia sets J_z is at least as important as the connectivity of J_2 to understanding the homology of the Fatou set.

In Section 7.2 we provide an example of a family of polynomial skew products F_a depending on a single complex parameter a with the following property: if a is in the Mandelbrot set \mathcal{M} the Fatou set $U(F_a)$ has trivial homology, while if a is outside of \mathcal{M} then $H_1(U(F_a))$ is infinitely generated.

Since neither of the sufficient conditions from Theorems 1.2 and 1.3 extend naturally to general endomorphisms, it remains a mystery what is an appropriate condition for endomorphism to have non-simply connected Fatou set. We conclude Section 7, and this paper, with a discussion of a few potential further applications of the techniques of this paper to holomorphic endomorphisms of \mathbb{P}^2 .

Acknowledgments. We thank John Hamal Hubbard for bringing us and some central ideas together at the start of this project, Laura DeMarco for feedback on the project and finding some appropriate references, Sarah Koch for discussions about her examples, Greg Buzzard and Eric Bedford for clarifying the current status on some of our ‘‘future questions’’, Enrique Pujals for a discussion resulting in major simplifications to Section 6, and Han Peters for a discussion of the examples in Section 7.2. The second author thanks Mikhail Lyubich and Ilia Binder for their financial support and mathematical guidance.

2. THE GREEN'S CURRENT T

In this section, we provide a brief reminder of the properties of the Green's current that will be needed later in this paper. We refer the reader who would like to see more details to [16, 27, 26]. While the following construction works more generally for generic (algebraically stable) rational maps (having points of indeterminacy), we restrict our attention to globally holomorphic maps of \mathbb{P}^k .

Suppose that $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ is holomorphic and that the Jacobian of f does not identically vanish on \mathbb{P}^k . Then f lifts to a polynomial map $F : \mathbb{C}^{k+1} \rightarrow \mathbb{C}^{k+1}$ each of whose coordinates

is a homogeneous polynomial of degree d . It is a theorem that

$$G(z) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|F^k(z)\|$$

converges to a plurisubharmonic² function $G : \mathbb{C}^{k+1} \rightarrow [-\infty, \infty)$ called the *Green's function associated to f* . Since f is globally well-defined on \mathbb{P}^k we have that $F^{-1}(0) = 0$. It has been established that G is Hölder continuous and locally bounded on $\mathbb{C}^{k+1} \setminus \{0\}$.

If $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ is the canonical projection, there is a unique positive closed $(1, 1)$ current T on \mathbb{P}^k satisfying $\pi^*T = \frac{1}{2\pi} dd^c G$. (This normalization is not uniform—many authors do not divide by 2π .) More explicitly, consider any open set $V \subset \mathbb{P}^k$ that is “small enough” so that a holomorphic section $\sigma : V \rightarrow \mathbb{C}^{k+1}$ of π exists. Then, on V we have that T is given by $T = \frac{1}{2\pi} dd^c(G \circ \sigma)$. Choosing appropriate open sets covering \mathbb{P}^k and sections of π on each of them, the result extends to all of \mathbb{P}^k producing a single closed positive $(1, 1)$ current on \mathbb{P}^k independent of the choice of open sets and sections used. See [26, Appendix A.4].

Recall that the Fatou set $U(f)$ is the maximal open set in \mathbb{P}^2 where the family of iterates $\{f^n\}$ form a normal family and that the Julia set of f is given by $J(f) = \mathbb{P}^k \setminus U(f)$. A major motivation for studying the Green's current is the following.

Theorem 2.1. *Let $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ be a holomorphic endomorphism and let T be the Green's current corresponding to f . Then, $J(f) = \text{supp } T$.*

See, for example, [27, Theorem 2.2] or [26, Theorem 3.3.2].

Remark 1. If f is a polynomial endomorphism, another form of Green's function, given by

$$G_{\text{affine}}(z) = \lim \frac{1}{d^n} \log_+ \|f^k(z)\|$$

is often considered in the literature. (Here $\log_+ = \max\{\log, 0\}$.) The result is again a plurisubharmonic function $G : \mathbb{C}^k \rightarrow [0, \infty)$.

We can relate G_{affine} to G in the following way. Consider the open set $V = \mathbb{C}^k \subset \mathbb{P}^k$. Using the section $\sigma(z_1, \dots, z_k) = (z_1, \dots, z_k, 1)$, we find $G_{\text{affine}}(z_1, \dots, z_k) = G \circ \sigma(z_1, \dots, z_k)$ because $\|F^k \circ \sigma\|$ only differs from $\|f^k\|$ by a bounded amount for each iterate k .

Therefore, if f is a polynomial endomorphism of \mathbb{P}^k , one can compute T on \mathbb{C}^k using the formula $T = \frac{1}{2\pi} dd^c G_{\text{affine}}$.

Remark 2. When $k = 1$, the resulting Green's current is precisely the measure of maximal entropy μ_f whose support is precisely the Julia set $J(f) \subset \mathbb{P}^1$. If f is a polynomial, then μ_f also coincides with the harmonic measure on $K(f)$ taken with respect to the point at infinity.

3. LINKING WITH THE GREEN'S CURRENT T .

In order to find non-trivial loops in the Fatou set we need a tool for relating the geometry of the Julia set $J_1 = \text{supp } T$ to the topology of the Fatou set $U(f) = \mathbb{P}^2 \setminus \text{supp } T$. In this situation, we show the technique of “linking” closed loops $\gamma \in U(f)$ with the Green's current T can be fruitful.

²We use the convention that plurisubharmonic functions cannot be identically equal to $-\infty$.

3.1. Linking numbers in \mathbb{S}^3 . Classically one considers the linking number of two oriented loops c and d in \mathbb{S}^3 . The linking number $lk(c, d) \in \mathbb{Z}$ is found by taking any oriented surface Γ with oriented boundary c and defining $lk(c, d)$ to be the signed intersection number of Γ with d as in Figure 3.1. For this and many equivalent definitions of linking number in \mathbb{S}^3 see [25, pp. 132-133], [5, pp. 229-239], and [23, Problems 13 and 14].

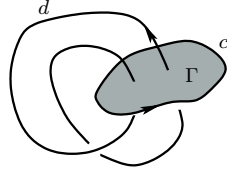


FIGURE 1. Here $lk(c, d) = +2$.

To see that this linking number is well-defined notice that assigning $lk(c, d) = [\Gamma] \cdot [d]$, where \cdot indicates the intersection product on $H_*(\mathbb{S}^3, c)$, coincides with the classical definition. (For background on the intersection product on homology, see [6, pages 366-372].) If Γ' is any other 2-chain with $\partial\Gamma' = c$ then $\partial(\Gamma - \Gamma') = [c] - [c] = 0$ and $(\Gamma - \Gamma')$ represents a homology class in $H_2(\mathbb{S}^3)$. Since $H_2(\mathbb{S}^3) = 0$, $[\Gamma - \Gamma'] = 0$ forcing $[\Gamma - \Gamma'] \cdot [d] = 0$. Therefore: $[\Gamma] \cdot [d] = [\Gamma'] \cdot [d]$, so that $lk(c, d)$ is well defined.

3.2. Linking with the Green's Current. Because closed currents appear naturally in complex dynamics, it is useful to extend the notion of linking numbers to a pairing between closed loops and closed currents. One approach is presented in [24] for the study of Newton's method, for which there are many points of indeterminacy. Here we develop a simplified approach for holomorphic maps that are globally defined on \mathbb{P}^2 .

Let M be a complex manifold, denote the space of degree q currents by $\mathcal{D}^q(M)$ and the subspace of closed currents by $Z^q(M)$. Let $C_i(M)$ denote the piecewise smooth i -dimensional chains on M .

Given any closed current $S \in Z^q(M)$ and any piecewise smooth q -chain σ with $\partial\sigma$ disjoint from $\text{supp } S$, we can define

$$\langle \sigma, S \rangle = \int_{\sigma} \eta_S$$

where η_S is a smooth approximation of S within its cohomology class in $\mathbb{P}^2 - \partial\sigma$, see [12, pages 382-385]. The resulting number $\langle \sigma, S \rangle$ will depend only on the cohomology class of S and the homology class of σ within $H_q(M, \partial\sigma)$.

We now return to the case $M = \mathbb{P}^2$. The following tentative definition is in direct analogy with the definition in \mathbb{S}^3 . Given any closed loop γ disjoint from $\text{supp } T$ (i.e. $\gamma \subset U(f)$), we could define

$$(1) \quad lk(\gamma, T) := \langle \Gamma, T \rangle$$

where Γ is any piecewise smooth two chain with $\partial\Gamma = \gamma$.

What made the linking numbers in \mathbb{S}^3 well-defined, independent of the choice of Γ , is the fact that $H_2(\mathbb{S}^3) = 0$. It is a well-known fact that $H_2(\mathbb{P}^2) \cong \mathbb{Z}$, generated by the fundamental class $[L]$ of any complex line L in \mathbb{P}^2 . This will result in the fact that the tentative definition for $lk(\gamma, T)$ given in (1) depends on the choice of Γ and hence is not well-defined.

For example if γ is in a complex line L one could choose Γ to be the “interior” of γ in L and Γ' to be the “exterior” of γ in L oriented appropriately so that $\partial\Gamma = \gamma = \partial\Gamma'$. Then as chains we have $\Gamma' = \Gamma + L$. This way, $\langle \Gamma', T \rangle = \langle \Gamma, T \rangle + \langle L, T \rangle \neq \langle \Gamma, T \rangle$, since $\langle L, T \rangle \neq 0$ for any line L .

One solution is to use that fact that the Green’s current T is normalized so that $\langle L, T \rangle = 1$ for any complex line L . (This is the reason for the normalization factor of 2π in the definition of T from Section 2). Given Γ and Γ' with $\partial\Gamma = \partial\Gamma' = \gamma$, $[\Gamma - \Gamma'] \in H_2(\mathbb{C}\mathbb{P}^2)$, so that $[\Gamma - \Gamma'] = k[L]$ for some integer k . Thus $\langle \Gamma, T \rangle - \langle \Gamma', T \rangle = \langle \Gamma - \Gamma', T \rangle = \langle kL, T \rangle = k\langle L, T \rangle = k$.

Therefore, $\langle \Gamma, T \rangle \pmod{1}$ is well-defined, independent of the choice of Γ with $\partial\Gamma = \gamma$.

Definition 3.1. *If γ is a piecewise smooth closed curve disjoint from the support of T , we define the linking number $lk(\gamma, T)$ by*

$$lk(\gamma, T) := \langle \Gamma, T \rangle \pmod{1}$$

where Γ is any piecewise smooth two chain with $\partial\Gamma = \gamma$.

Unlike linking numbers between closed loops in \mathbb{S}^3 , it is often the case that that $\langle \Gamma, T \rangle \notin \mathbb{Z}$, resulting in non-zero linking numbers $\pmod{1}$.

Proposition 3.2. *Suppose that γ_1 and γ_2 are homologous in $H_1(\mathbb{P}^2 \setminus \text{supp } T)$, then $lk(\gamma_1, T) = lk(\gamma_2, T)$.*

Proof. Let Γ be any piecewise smooth two chain contained in $\mathbb{P}^2 \setminus \text{supp } T$ with $\partial\Gamma = \gamma_1 - \gamma_2$. Then, since $\mathbb{P}^2 \setminus \text{supp } T$ is open and Γ is compact subset, Γ is bounded away from the support of T . Consequently for any smooth approximation η_T of T supported in a sufficiently small neighborhood of T , we have $lk(\gamma_1, T) - lk(\gamma_2, T) = \int_{\Gamma} \eta_T = 0$. \square

Recall that for holomorphic endomorphisms $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$, the Fatou set $U(f)$ of f coincides with $\mathbb{P}^2 \setminus \text{supp } T$, so that as a corollary we have

Theorem 3.3. *If γ is a smooth loop in $U(f)$ with $lk(\gamma, T) \neq 0$, then $[\gamma] \neq 0 \in H_1(U(f))$.*

Hence, a non-trivial linking number with the Green’s current T can be used to show that a smooth loop γ in the Fatou set has non-trivial homology. This is the first step in establishing Theorem 1.1.

3.3. Properties of the pairing $\langle \cdot, \cdot \rangle$ and restriction to slices. We now prove two important properties of the pairing $\langle \cdot, \cdot \rangle$, which will be useful when computing linking numbers. While we will use them for the Green’s current T , we consider the case of an arbitrary positive closed $(1, 1)$ current λ .

We recall the definition of pull-back for closed-positive $(1, 1)$ currents under ramified mappings, as presented in [26, Appendix A.7] and [16, p. 330-331].

Suppose that $F : \Omega \rightarrow \Omega'$ is a (possibly ramified) analytic mapping between open sets in \mathbb{C}^m with the Jacobian of F not identically 0. If λ is any closed positive $(1, 1)$ current on Ω' given by $\lambda = dd^c u$, then λ can be pulled back under F by pulling back the potential: $F^*(\lambda) := dd^c(u \circ F)$. The hypothesis that the Jacobian of F is not identically 0 ensures $F(\Omega)$ contains an open set, and is consequently not contained the polar locus of u . In this case, $u \circ F$ is not identically equal to $-\infty$ and hence $dd^c(u \circ F)$ gives a well-defined positive closed $(1, 1)$ current.

We will find it useful to restrict a positive closed current λ on M to an analytically embedded submanifold N of lower dimension. If $\phi : N \rightarrow M$ is the embedding, the restriction

should be given by $\lambda|_N = \phi^*\lambda = dd^c(u \circ \phi)$. However, since $\phi(N)$ does not contain an open set in M , it is possible that $\phi(N)$ is contained in the polar locus of u , so that $u \circ \phi \equiv -\infty$, in which case $dd^c u \circ \phi$ is not defined. This happens in very simple cases, for instance when if $\phi(N)$ is a complex line in \mathbb{P}^2 , and λ is the current of integration over that line.

Since any two potentials for λ differ by a pluriharmonic function, they have the same polar locus, which we will refer to as the polar locus of λ . In the case that $\phi(N)$ is not in the polar locus of λ , then the restriction $\lambda|_N = \phi^*\lambda = dd^c(u \circ \phi)$ is defined. (The condition that $\phi(N)$ is not in the polar locus of λ can be thought of as a weak form of transversality between $\phi(N)$ and the current λ .)

Proposition 3.4. *If λ is a positive closed $(1,1)$ current on M' and $F : M \rightarrow M'$ is an analytic mapping with $F(M)$ not entirely contained in the polar locus of λ , then we have $\langle F_*\sigma, \lambda \rangle = \langle \sigma, F^*\lambda \rangle$.*

Proof. Let η_λ be a smooth approximation of λ in the same cohomology class. Then, $\langle F_*\sigma, \lambda \rangle = \int_{F_*\sigma} \eta_\lambda = \int_\sigma F^*\eta_\lambda = \langle \sigma, F^*\lambda \rangle$, since $F^*\eta_\lambda$ is a smooth approximation of $F^*\lambda$.

By hypothesis, the pull-back $F^*\lambda$ is well-defined, so that right hand side of the last equality is well-defined. \square

We only need restrictions when $M = \mathbb{P}^2$, so we prove the following:

Corollary 3.5. *Let λ be a positive closed $(1,1)$ current on \mathbb{P}^2 and $\phi : \sigma \rightarrow \mathbb{P}^2$ an analytically embedded curve in \mathbb{P}^2 . If $\phi(\sigma)$ is not contained in the polar locus of λ , then $\langle \phi(\sigma), \lambda \rangle = \int_\sigma \lambda|_\sigma$, where $\lambda|_\sigma$ is the restriction of λ to σ .*

This restriction $\lambda|_\sigma$ is a positive measure on σ which in terms of local coordinate $z = x+iy$ is given by the Laplacian: $\Delta(u \circ \phi)dx \wedge dy$.

Proof. The restriction $\lambda|_\sigma$ is given by $\phi^*(\lambda)$. Then, $\langle \phi(\sigma), \lambda \rangle = \langle \sigma, \phi^*\lambda \rangle = \langle \sigma, \lambda|_\sigma \rangle$ follows from Proposition 3.4. Since $\lambda|_\sigma$ is a positive measure, $\int_\sigma \lambda|_\sigma$ is well-defined, and coincides with the result obtained by first choosing a smooth approximation to $\lambda|_\sigma$.

On a complex curve dd^c reduces to the Laplacian: $dd^c f = \Delta f dx \wedge dy$, see [16]. Thus, if u is a potential for λ , in suitable local coordinates we have $\lambda|_\sigma = dd^c(u \circ \phi) = \Delta(u \circ \phi)dx \wedge dy$, where Δ is the Laplacian in the sense of distributions. \square

Remark 3. It is well known that the Green's current T for an endomorphism has empty polar locus so that the result of Corollary 3.5 can be applied to any analytic curve when $\lambda = T$. See [16, Theorem 2.1] and [26, Theorem 1.7.1].

In fact, if σ has no singular points, the resulting measure coincides with the wedge product $T \wedge [\phi(\sigma)]$, where $[\phi(\sigma)]$ is the current of integration over the image $\phi(\sigma) \subset \mathbb{P}^2$.

Remark 4. When computing linking numbers we will often choose a piecewise smooth loop γ within some complex projective line L . In this case, either of the two complementary regions to γ within L are natural choices for Γ . Then, if $\int_\Gamma T|_L \neq 0, 1$, a condition that is often easy to check, then γ has nontrivial linking number with T in \mathbb{P}^2 .

3.4. Example. Consider the polynomial skew product $(z, w) \mapsto (z^2, w^2 + 0.3z)$, for which the Fatou set consists of the union of basins of attraction for three super-attracting fixed points: $[0 : 1 : 0]$, $[0 : 0 : 1]$, and $[1 : 0 : 0]$. In Figure 2 we show a computer generated image of the intersection of $W^s([0 : 1 : 0])$ (lighter gray) and $W^s([0 : 0 : 1])$ (dark gray) with the vertical line $z = z_0 = 0.99999$. In terms of the fiber-wise Julia sets that were mentioned in the introduction, K_{z_0} is precisely the closure of the dark gray region and J_{z_0} is its boundary.

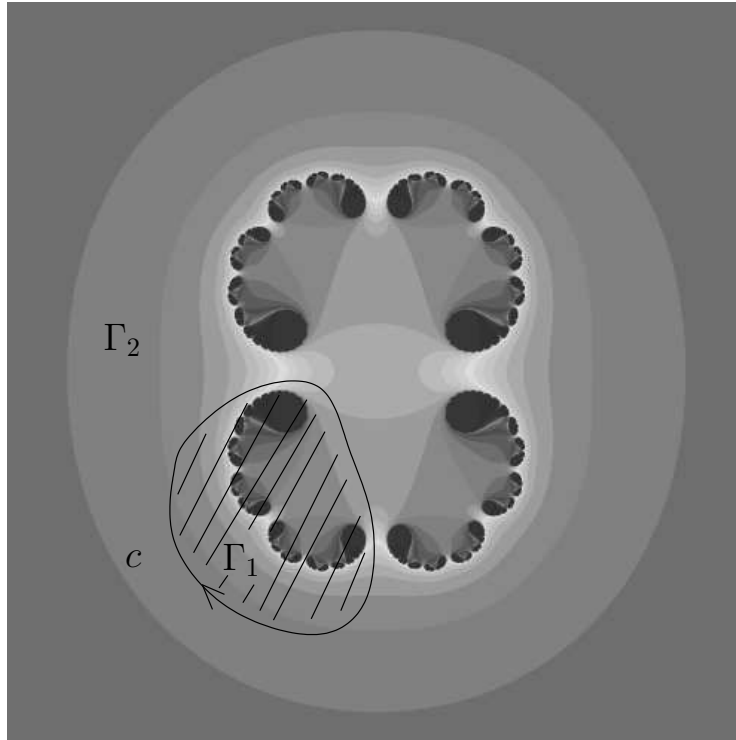


FIGURE 2. Both choices Γ_1 (inside of γ) and Γ_2 (outside of γ) yield the same $lk(c, T)$.

We will see in Proposition 6.2 that $T|_{z=z_0}$ is precisely the harmonic measure on K_{z_0} . Using this knowledge, and supposing that the computer image is accurate, we illustrate how the above definitions can be used to show that the smooth loop c shown in the figure represents a non-trivial homology class in $H_1(W^s([0 : 1 : 0]))$.

Suppose that we use two chain Γ_1 that is depicted in the figure to compute $lk(c, T)$. The harmonic measure on K_{z_0} is supported in J_{z_0} and equally distributed between the four symmetric pieces with total measure of K_{z_0} is 1. Therefore (using Corollary 3.5) we see that $lk(c, T) = \int_{\Gamma_1} T|_{z=z_0} = \frac{1}{4} \pmod{1}$, because Γ_1 covers exactly 1 these 4 pieces of K_{z_0} .

If instead we use Γ_2 , the disc “outside of c ” within the projective line $z = z_0$ with the orientation chosen so that $\partial\Gamma_2 = c$ as depicted, then $lk(c, T) = \int_{\Gamma_2} T|_{z=z_0} = -\frac{3}{4} \pmod{1}$ (because Γ_2 covers 3 of the 4 symmetric pieces of K_{z_0} , but with the opposite orientation than that of Γ_1). However, $-\frac{3}{4} \pmod{1} = \frac{1}{4} \pmod{1}$, so we see that the computed linking number does come out the same.

Since $lk(c, T) \neq 0 \pmod{1}$, Corollary 3.3 gives that it is impossible to have any 2-chain Λ within $W^s([0 : 1 : 0])$ (even outside of the vertical line $z = z_0$) so that $\partial\Lambda = c$. Thus $[c] \neq 0 \in H_1(W^s([0 : 1 : 0]))$.

4. INFINITELY GENERATED HOMOLOGY

Given $\gamma \subset U(f)$ with $[\gamma]$ non-trivial, it is natural to try to use the iterates of f to lift γ in order to show that not only is $H_1(U(f))$ non-trivial, but that $H_1(U(f))$ is infinitely generated.

We consider that possibility when γ is contained within the basin of attraction $W^s(\zeta)$ of an attracting (or super-attracting) fixed point ζ for a holomorphic map $f : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$. Appropriately stated analogous results also hold for attracting periodic points.

4.1. Degree of nilpotence. We now consider the first homology $H_1(W^s(\zeta))$. In this paper the coefficient ring will always be \mathbb{Z} . For background on homology theory, we recommend to the reader [13, 6]. Suppose that ζ is an attracting fixed point for holomorphic $f : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$ and that γ is some closed loop (or more generally 1-cycle) with $\gamma \subset W^s(\zeta)$. Since ζ is attracting (or super-attracting), there is a small topological ball U containing ζ which is entirely contained in $W^s(\zeta)$. Since γ is compactly contained in $W^s(\zeta)$ some iterate $f^k(\gamma)$ will be contained in U , hence $[f^k(\gamma)] \in H_1(W^s(\zeta))$ will be trivial. The smallest k for which this occurs has special significance.

Definition 4.1. *The degree of nilpotence $\mathcal{N} : H_1(W^s(\zeta)) \rightarrow \mathbb{Z}^+$ is $\mathcal{N}(\sigma) = k$ where k is first iterate for which $f_*^k(\sigma) = 0$.*

A very crucial property of \mathcal{N} is the following form of linearity:

$$(2) \quad \mathcal{N}(a_1\sigma_1 + \cdots + a_n\sigma_n) \leq \max(\mathcal{N}(\sigma_1), \dots, \mathcal{N}(\sigma_n)).$$

which follows immediately from linearity of the action of f_* .

Proposition 4.2. *Suppose that ζ is an attracting fixed point of a holomorphic map $f : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$. If there are elements $\sigma \in H_1(W^s(\zeta))$ with arbitrarily high degree of nilpotence $\mathcal{N}(\sigma)$, then $H_1(W^s(\zeta))$ is an infinitely generated group (module.)*

Proof. The proof follows immediately from the linearity property given in Equation 2. Suppose that $H_1(W^s(\zeta))$ is generated by $\sigma_1, \dots, \sigma_n$. Then any $\sigma \in H_1(W^s(\zeta))$ has $\sigma = a_1\sigma_1 + \cdots + a_n\sigma_n$ and Equation 2 gives that $\mathcal{N}(\sigma) \leq \max(\mathcal{N}(\sigma_1), \dots, \mathcal{N}(\sigma_n))$, so that every σ has bounded degree of nilpotence. \square

4.2. Lifting loops. In the case that $H_1(W^s(\zeta)) \neq 0$, the following is a straightforward strategy for lifting non-trivial loops in $W^s(\zeta)$ under the dynamics of f in order to prove that $H_1(W^s(\zeta))$ is infinitely generated. Unfortunately, it does not directly give the desired result.

Any $\sigma \neq 0 \in H_1(W^s(\zeta))$ can be represented by a smooth closed loop $\gamma \subset W^s(\zeta)$. The critical value locus of f has real dimension 2 and γ has real dimension 1, so since $W^s(\zeta)$ is an open set, we can perturb γ within its homology class so that γ is disjoint from the critical value locus. Then, $f^{-1}(\gamma) := \eta_1 \cup \cdots \cup \eta_m$, with each η_i a closed loop in $W^s(\zeta)$ and $f : \eta_i \rightarrow \gamma$ a covering map of some degree d_i , and $d_1 + \cdots + d_m = d^k = \deg_{\text{top}}(f)$.

Since $f_*([\eta_i]) = d_i[\gamma]$, if $d_i[\gamma] \neq 0$, then $[\eta_i] \neq 0$ giving $\mathcal{N}(\eta_i) = \mathcal{N}(\gamma) + 1$. This happens, in particular, if $[\gamma]$ is not *torsion*, meaning that there is no $l \in \mathbb{Z}$ with $l[\gamma] = 0$. However, if $[\gamma]$ is torsion, so that $l[\gamma] = 0$ for some $l \in \mathbb{Z}$, it may be possible that d_i is a multiple of l , in which case one cannot conclude that $[\eta_i] \neq 0$.

Question 4.3. If $f : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$ is holomorphic and ζ is an attracting fixed point (or periodic point), can there be torsion elements of $H_1(W^s(\zeta))$?

Since the authors do not know the answer to this question, rather than use the above strategy alone, we use it in combination with the degree of nilpotence as stated in the following corollary to Proposition 4.2.

Corollary 4.4. *If $H_1(W^s(\zeta)) \neq 0$ and there is some $[\gamma] \neq 0 \in H_1(W^s(\zeta))$ which has a sequence of lifts $[\eta^1], \dots, [\eta^k], \dots$ under f so that $[\eta^i] \neq 0$ for each i , then $H_1(W^s(\zeta))$ is infinitely generated.*

This corollary in combination with Theorem 3.3 establishes Theorem 1.1.

5. POLYNOMIAL ENDOMORPHISMS OF \mathbb{P}^2

In this section we establish Theorem 1.2, providing our first application of Theorem 1.1.

Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be given by $f(z, w) = (p(z, w), q(z, w))$ for some polynomials p and q . Suppose that p and q are both of degree d and that $p_d(z, w)$ and $q_d(z, w)$ are the homogeneous polynomials consisting of the degree d terms from p and from q . If p_d and q_d satisfy that $p_d(z, w) = 0 = q_d(z, w)$ only if $z = w = 0$ then f extends to holomorphic map on \mathbb{P}^2 given in homogeneous coordinates by

$$(3) \quad f([Z : W : T]) = [P(Z, W, T) : Q(Z, W, T) : T^d],$$

where P and Q are the homogenizations of p and q . We call any endomorphism of \mathbb{P}^2 that can be obtained in this way a *polynomial endomorphism* of \mathbb{P}^2 .

The (projective) line at infinity $\Pi := \{T = 0\}$ is uniformly superattracting for any polynomial endomorphism f . The restriction f_Π is given in homogeneous coordinates by

$$(4) \quad f_\Pi : ([Z : W]) \rightarrow [p_d(Z, W) : q_d(Z, W)].$$

where p_d and q_d are the degree d terms from p and q , as mentioned above. We will denote the lift of f_Π to \mathbb{C}^2 by F_Π .

Let $U(f)$ be the Fatou set for f and $U(f_\Pi)$ the Fatou set for f_Π . The former is an open set in \mathbb{P}^2 , while the latter is an open set in the line at infinity Π .

Lemma 5.1. *If f_Π is hyperbolic then $U(f_\Pi) \subset U(f)$.*

Proof. Suppose that f_Π is hyperbolic with non-empty Fatou set $U(f_\Pi)$. Let $x \in U(f_\Pi)$. Since f_Π is hyperbolic, each Fatou component is the basin of attraction of an attracting (or, possibly super-attracting) periodic cycle. We suppose x is in the basin of attraction of ζ_1, \dots, ζ_n . Since the line at infinity Π is transversally superattracting, the cycle ζ_1, \dots, ζ_n is superattracting in the transverse direction to Π and (at least) geometrically attracting within Π . Thus, it is attracting in \mathbb{P}^2 under f . As such, x is in the basin of attraction for an attracting cycle of f , and hence within $U(f)$. \square

Let T be the Green's current for f and let μ_Π be the measure of maximal entropy for the restriction f_Π .

Lemma 5.2. *The restriction $T|_\Pi$ coincides with μ_Π .*

Proof. Consider the lift $F_\Pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of the rational map $f_\Pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. As observed in Remark 2,

$$G_\Pi(Z, W) = \lim \frac{1}{d^n} \log \|F_\Pi^n(Z, W)\|$$

is the potential for μ_Π , meaning that $\pi^* \mu_\Pi = \frac{1}{2\pi} dd^c G_\Pi$.

The restriction $T|_\Pi$ is obtained by restricting of the potential G to $\pi^{-1}(\Pi) = \{(Z, W, 0) \in \mathbb{C}^3\}$. Specifically, it is defined by $\pi^*(T|_\Pi) = \frac{1}{2\pi} dd^c(G(Z, W, 0))$. Therefore, it suffices to show that $G(Z, W, 0) = G_\Pi(Z, W)$. However, this follows directly from the fact that $F(Z, W, 0) =$

$F_{\Pi}(Z, W)$. (Here F is the lift of f to \mathbb{C}^3 , as given by (3) when considered in non-projective coordinates $[Z, W, T]$.) \square

We now establish Theorem 1.2: that if f_{Π} is hyperbolic and J_{π} is disconnected, then the Fatou set $U(f)$ has infinitely generated first homology.

Proof of Theorem 1.2. Since J_{Π} is disconnected, there is some piecewise smooth simple closed curve γ in the Fatou set $U(f_{\Pi})$ for f_{Π} that disconnects J_{Π} . Since $U(f_{\Pi}) \subset U(f)$ $\gamma \in U(f)$, so that the linking number $lk(\gamma, T)$ is well-defined.

Let Γ be one of the two closed discs in Π oriented so that $\partial\Gamma = \gamma$ in the sense of chains. Because the support of the measure μ_{Π} equals J_{Π} with γ disconnecting J_{Π} , we necessarily have

$$0 < \left| \int_{\Gamma} \mu_{\Pi} \right| < 1.$$

(The absolute value is taken in the case that Γ has the opposite orientation of Π .)

By Lemma 5.2 we have that $T|_{\Pi} = \mu_{\Pi}$ so that applying Corollary 3.5 we have that $\langle \Gamma, T \rangle = \int_{\Gamma} T|_{\Pi}$ also satisfies

$$0 < |\langle \Gamma, T \rangle| < 1.$$

This gives that $lk(\gamma, T) = \langle \Gamma, T \rangle \pmod{1} \neq 0$. Thus, by Theorem 3.3 we have that γ represents a non-trivial homology class in $H_1(U(f))$.

Now we show that $H_1(U(f))$ is actually infinitely generated. In order to use Corollary 4.4, we will show that there is a sequence of lifts $\gamma_1, \gamma_2, \dots$ of γ , with each of the γ_i represents a non-trivial homology class. We do the construction entirely within the line Π . The reason for this is that $f_{\Pi} : \Pi \rightarrow \Pi$ is a branched covering of degree d which exactly coincides with the factor that multiplies T under pull-back: $f^*(T) = d \cdot T$.

Each time that a curve γ_{i-1} is lifted, it is perturbed away from the (finite number) of critical values of f_{Π} so that the resulting lift γ_i is again a (closed) curve. Because γ_i lies in the complex line Π and has no crossings, it is a simple closed curve. Let Γ_i be the closed disc in Π having $\partial\Gamma_i = \gamma_i$ with the property that $f_{\Pi}^i(\Gamma_i)$ covers Γ . (Recall that Γ was the disc bounding the initial curve γ .)

Since $f_{\Pi} : \Pi \rightarrow \Pi$ is a ramified covering of degree d , $f_{\Pi}^i : \Gamma_i \rightarrow \Gamma$ is a ramified cover of degree $0 < l \leq d^i$. Using Proposition 3.4 we find that

$$d^i \langle \Gamma_i, T \rangle = \langle \Gamma_i, (f^i)^*T \rangle = \langle f^i(\Gamma_i), T \rangle = l \langle \Gamma, T \rangle$$

Which gives

$$|\langle \Gamma_i, T \rangle| = \frac{l}{d^i} |\langle \Gamma, T \rangle|.$$

Thus, $0 < |\langle \Gamma_i, T \rangle| < 1$, giving $lk(\gamma_i, T) = \langle \Gamma_i, T \rangle \pmod{1} \neq 0$.

Therefore, we have an infinite sequence of lifts $\gamma_1, \gamma_2, \dots$ of γ each of which represents a non-trivial homology class in $H_1(U(f))$. Corollary 4.4 then gives that $H_1(U(f))$ is infinitely generated. \square

There are many polynomial endomorphism to which Theorem 1.2 applies. We provide one family of examples.

Example 5.3. Consider the cubic polynomial $r(z) = z^3 - 0.48z + (0.706260 + 0.502896i)$. One of the critical points for this polynomial escapes to infinity, while the other is in the basin of attraction for a cycle of period 3. The result is a filled Julia set with infinitely many non-trivial connected components, each of which is homeomorphic to the Douady's rabbit. (See [22].)

We embed this as the dynamics on the line at infinity Π for a polynomial endomorphism of \mathbb{P}^2 . Let $R(z, w) = z^3 - 0.48zw^2 + (0.706260 + 0.502896i)w^3$ be the homogeneous form of r , and let $p(z, w, t)$ and $q(z, w, t)$ be any homogeneous polynomials of degree 2. Then

$$f([z : w : t]) = [R(z, w) + tp(z, w, t) : w^3 + tq(z, w, t) : t^3]$$

is a polynomial endomorphism with $f_\Pi = r$. In this case, Theorem 1.2 gives that the basin of attraction of $[1 : 0 : 0]$ for f has infinitely generated first homology.

6. POLYNOMIAL SKEW PRODUCTS

Polynomial skew products are a special type of polynomial endomorphism of \mathbb{P}^2 . A polynomial skew product is a map $f(z, w) = (p(z), q(z, w))$ with p and q polynomials of degree d where $p(z) = z^d + O(z^{d-1})$ and $q(z) = w^d + O_z(w^{d-1})$. This assumption is necessary in order for f to extend to a holomorphic endomorphism of \mathbb{P}^2 , on which we write homogeneous coordinates as $[Z : W : T]$ with (z, w) corresponding to $[z : w : 1]$.

There are many polynomial skew products f for which Theorem 1.2 readily gives that $H_1(U(f))$ is infinitely generated; for example, $f(z, w) = (z^2, w^2 + 10z^2)$ has dynamics on Π given by $w \rightarrow w^2 + 10$. Next we will find alternative sufficient conditions under which a polynomial skew product has Fatou set with infinitely generated first homology, proving Theorem 1.3. This theorem applies to many maps to which Theorem 1.2 does not apply; for example, $f(z, w) = (z^2, w^2 - 3z)$, for which J_Π is equal to the unit circle.

6.1. Preliminary results on polynomial skew products. We refer the reader to Jonsson [19] for a general treatment of skew products.

In order to apply Theorem 1.1 to skew products, we first observe that every polynomial skew product f has $[0 : 1 : 0]$ as a super-attracting fixed point which is totally invariant under f . In Section 6.3, we will apply Corollary 4.4 to $\zeta = [0 : 1 : 0]$, showing that the basin of attraction $W^s([0 : 1 : 0])$ for certain polynomial skew products has $H_1(W^s([0 : 1 : 0]))$ infinitely generated.

It is interesting to observe that:

Proposition 6.1. *If ζ is a totally-invariant (super)-attracting fixed point for a holomorphic $f : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$, then $W^s(\zeta)$ is path connected.*

A nearly identical statement is proven for $\mathbb{C}\mathbb{P}^2$ in Theorem 1.5.9 from [17]. We refer the reader to their proof since the proof is nearly identical for $\mathbb{C}\mathbb{P}^k$.

There is a very convenient description of the fiber-wise dynamics in terms of the Green's current. For $f(z, w) = (p(z), q(z, w))$ a polynomial skew product, let G_{affine} be the (affine) Green's function of f , and T the Green's current. Since f preserves the family of vertical lines, $\{z\} \times \mathbb{C}$, it is useful to examine f via the dynamics of the base map p , combined with the fiber maps $q_z(w) = q(z, w)$. Recall $(K_p)J_p, G_p$ denote the (filled) Julia set and Green's function for the base map p , respectively. We also utilize the fiberwise Green's functions $G_z(w) := G_{\text{affine}}(z, w) - G_p(z)$, and Julia sets $K_z := \{G_z = 0\}, J_z := \partial K_z$.

The global filled Julia set, K , is the zero set of G_{affine} , i.e., points in \mathbb{C}^2 with bounded orbits. If $z \in K_p$, an easy consequence is that $K_z = K \cap (\{z\} \times \mathbb{P})$.

As mentioned in Section 3.3 we can restrict the current T to any analytic curve obtaining a measure on that curve. Of particular interest for skew products is the restriction μ_{z_0} of T to a vertical line $(\{z_0\} \times \mathbb{P})$. The following appears as Jonsson [19] Proposition 2.1 (i), we repeat it here for completeness:

Proposition 6.2. *The restriction $T|_{z=z_0}$ of the Green's current T to a vertical line $(\{z_0\} \times \mathbb{P})$ coincides with the harmonic measure μ_{z_0} on K_{z_0} .*

Proof. Notice that

$$\begin{aligned} T|_{z=z_0} &= \frac{1}{2\pi} dd^c G|_{z=z_0} = \frac{1}{2\pi} dd^c G(z_0, w) \\ &= \frac{1}{2\pi} dd^c (G(z_0, w) - G_p(z_0)) = \frac{1}{2\pi} dd^c G_{z_0}(w). \end{aligned}$$

According to [19, Thm 2.1], G_{z_0} is the Green's function for K_z with pole at infinity. We have thus obtained that μ_{z_0} is exactly the harmonic measure μ_{z_0} on K_{z_0} . \square

There is also (as in the case of any endomorphism) the measure of maximal entropy $\mu = T \wedge T$. By Jonsson [19], $J_2 := \text{supp}(\mu) = \overline{\bigcup_{z \in J_p} J_z}$ is also the closure of the set of repelling periodic points.

6.2. Control of $\text{supp } T$ for polynomial skew products. Next, we provide a lemma on connectivity of the fiber Julia sets in a neighborhood of J_p , which gives us the information on the geometry of $\text{supp } T$ that we need for establishing Theorem 1.3.

Lemma 6.3. *Suppose f is a polynomial skew product. If for some $z \in J_p$, J_z is disconnected, then there is a neighborhood $U \subset \mathbb{C}$ of z such that for all $x \in U$, J_x is disconnected.*

Proof. Fix z such that $z \in J_p$ and J_z is disconnected. Suppose to the contrary that for any $n \in \mathbb{N}$, there is an $x_n \in \mathbb{D}(z, 1/n) \subset \mathbb{C}$ (the disk about z of radius $1/n$) such that J_{x_n} is connected. Note by their definitions that for any $x \in \mathbb{C}$, J_x is disconnected iff K_x is disconnected, hence each K_{x_n} is connected. For any fixed x in \mathbb{C} , the fiber (or vertical) critical points are defined by $C_x := \{w : q'_x(w) = 0\}$. Jonsson shows ([19, Proposition 2.3]) that if f is any polynomial skew product on \mathbb{C}^2 , and $x \in \mathbb{C}$, then J_x is connected if and only if $C_{p^n(x)} \subset K_{p^n(x)}$ for all $n \geq 0$. Hence, $C_{x_n} \subset K_{x_n}$ for all n . But note that as a set, C_z is fiberwise continuous (i.e., $z \rightarrow C_z$ is continuous), since it is an analytically defined locus. So $C_{x_n} \rightarrow C_z$ as $n \rightarrow \infty$. But since by [19, Prop. 2.1] K_z varies upper semi-continuously, we must have $C_z \subset K_z$. From this it follows that $C_{p^n(z)} \subset K_{p^n(z)}$, for example since $G_{p(x)} \circ q_x = dG_x$ for any $x \in \mathbb{C}$. Hence K_z is connected. This is a contradiction. \square

6.3. Non-simply connected skew products. The following theorem implies Theorem 1.3.

Theorem 6.4. *Suppose f is a polynomial skew product and that for some $z \in J_p$, J_z is disconnected. Then the basin of attraction $W^s([0 : 1 : 0])$ has infinitely generated first homology.*

Proof. We first show that $H_1(W^s([0 : 1 : 0])) \neq 0$ by finding a specific loop $\gamma \subset W^s([0 : 1 : 0])$ representing a non-trivial class. We then check the hypotheses of Corollary 4.4 are satisfied for this γ , in order to see that $H_1(W^s([0 : 1 : 0]))$ is infinitely generated.

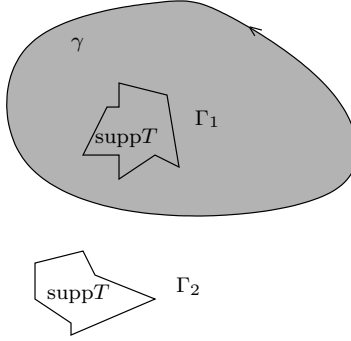


FIGURE 3. Γ_1 and Γ_2 are the closures of the two discs complementary to γ .

Step 1: $H_1(W^s([0 : 1 : 0]))$ is non-trivial:

By hypothesis, there is some $z \in J_p$ so that J_z is disconnected. Then, by Lemma 6.3, there is some neighborhood $U \subset \mathbb{C}$ of z so that for every $z_0 \in U$, J_{z_0} is disconnected. Since p is a polynomial, J_p has no interior, so there is some $z_0 \in U \setminus J_p$. Since J_p is closed, there is a neighborhood $U' \subset U$ of z_0 with $U' \cap J_p = \emptyset$.

Furthermore, since U' is open and disjoint from J_p , we can also assume that z_0 is not postcritical for p .

Consider now the vertical line $\{z_0\} \times \mathbb{P}$. Since J_{z_0} is disconnected, so is K_{z_0} and we can choose a piecewise smooth simple closed loop $\gamma \subset \{z_0\} \times (\mathbb{P} \setminus K_{z_0})$ disconnecting K_{z_0} . Since γ is in the complement of K_{z_0} , we have $\gamma \subset W^s([0 : 1 : 0])$, by definition of the fiber filled Julia sets ($K_x = \{G_x = 0\}$).

Since γ is a compact subset of the open set $W^s([0 : 1 : 0])$, which is certainly in the Fatou set, we find that γ is bounded away from $\text{supp}(T)$. Thus the linking number $lk(\gamma, T)$ is a well defined invariant of $[\gamma]$ (modulo 1) and be computed using any piecewise smooth two chain Γ with $\partial\Gamma = \gamma$.

The Jordan curve theorem gives that $(\{z_0\} \times \mathbb{C}) \setminus \gamma$ is the union of two open discs D_1, D_2 . Let $\Gamma_i = D_i \cup \gamma$ for $i = 1, 2$. See Figure 3.

Choosing orientations for Γ_i appropriately we have $\partial\Gamma_i = \gamma$. Since Γ_i is an analytically embedded 2 chain, Corollary 3.5 gives $lk(\gamma, T) = \langle \Gamma_1, T \rangle = \int_{\Gamma_1} T|_{\Gamma_1}$.

By Proposition 6.2 when we restricting T to this line we get $T|_{\{z_0\} \times \mathbb{P}} = \mu_{z_0}$, where μ_{z_0} is harmonic measure on K_{z_0} . Thus, $\text{supp}(\mu_{z_0}) = J_{z_0}$. Since γ disconnects K_{z_0} , γ disconnects $J_{z_0} = \text{supp}(\mu_{z_0})$. The total mass of μ_{z_0} is 1 with $\text{supp}(\mu_z) \cap \Gamma_i \neq \emptyset$ for $i = 1, 2$. Consequently,

$$(5) \quad 0 < |\langle \Gamma_i, T \rangle| = \left| \int_{\Gamma_i} \mu_{z_0} \right| < 1$$

for $i = 1, 2$. (We use the absolute value, since Γ_i will have negative orientation for $i = 1$ or $i = 2$.)

Thus $lk(\gamma, T) = \int_{\Gamma_1} \mu_{z_0} \neq 0 \pmod{1}$ gives that the homology class of γ is nontrivial: $[\gamma] \neq 0$.

Step 2: $H_1(W^s([0 : 1 : 0]))$ is infinitely generated:

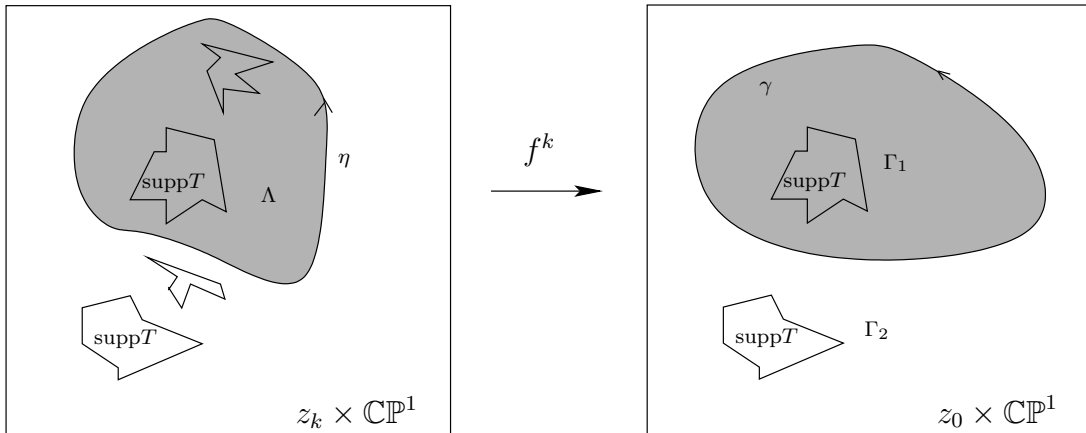


FIGURE 4. Illustration of Λ , gray region on left in the case that $f^k(\Lambda) = \Gamma_1$, shown as the gray region on the right. Support of T is shown schematically as polygonal curves.

We will now show that the specific γ obtained in the previous part satisfies the hypotheses of Corollary 4.4 in order to show that $H_1(W^s([0 : 1 : 0]))$ is infinitely generated.

Recall that we chose z_0 not in the postcritical set for p . We use this fact to show that for each k , there are “sufficiently many distinct lifts” η_1, \dots, η_m of γ under f^k so that at least one of them has to be non-trivial.

For any k , the critical value locus of f^k intersects $\{z_0\} \times \mathbb{P}^1$ in a finite number of points. This follows because z_0 is not postcritical for p , hence not within a critical value for p^k and because the remaining critical value curves for f intersect every vertical line transversally. Therefore, moving γ within $(\{z_0\} \times \mathbb{P}^1) \cap W^s([0 : 1 : 0])$ we can ensure that γ is disjoint from the postcritical set of f^k . This allows us to lift a curve entirely within $\{z_0\} \times \mathbb{P}^1$, so that the lifts η_1, \dots, η_m of γ under f^k lie in vertical lines over $\{p^{-k}(z_0)\}$.

Since z_0 is not postcritical for p , $\{p^{-k}(z_0)\}$ consists of precisely d^k points. Hence $f^{-k}(\{z_0\} \times \mathbb{P}^1)$ consists of precisely d^k vertical lines each of which contains at least one lift η_i of γ under f^k . Therefore, there are at least d^k disjoint lifts η_1, \dots, η_m of γ under f^k .

Denote by d_i the degree of covering for $f : \eta_i \rightarrow \gamma$, so that $d_1 + \dots + d_m = (d^2)^k = (\deg_{\text{top}}(f))^k$. Since $m \geq d^k$, we can therefore conclude that at least one of the d_i has $d_i \leq d^k$. For ease of notation, we let $\eta := \eta_i$ and $\delta := d_i$.

We now observe that $lk(\eta, T) \neq 0 \pmod{1}$. Since η is a simple closed curve in some vertical line $\{z_k\} \times \mathbb{P}^1$, we can find a two chain Λ with $\partial\Lambda = \eta$ by choosing the closure of either of the two topological discs in $(\{z_k\} \times \mathbb{P}^1) \setminus \eta$, with the proper orientation. Then, $f : (\{z_k\} \times \mathbb{P}^1) \rightarrow (\{z_0\} \times \mathbb{P}^1)$ is a branched covering, because f is a skew product, and within this line $f : \Lambda \rightarrow \Gamma_i$ is a branched covering of degree δ . Here, Γ_i is the closure of one of the two discs in $(\{z_0\} \times \mathbb{P}^1) \setminus \gamma$, as constructed in the first half of the proof. For ease of notation, drop the index i , and just write Γ . Thus, as 2-chains, $f(\Lambda) = \delta \cdot \Gamma$. Figure 4 provides an illustration.

We now use Proposition 3.4 and the fact that the Green’s current T is invariant: $(f^k)^*(T) = d^k T$:

$$d^k \langle \Lambda, T \rangle = \langle \Lambda, (f^k)^* T \rangle = \langle f^k(\Lambda), T \rangle = \delta \cdot \langle \Gamma, T \rangle$$

Consequently,

$$(6) \quad \langle \Lambda, T \rangle = \frac{\delta}{d^k} \cdot \langle \Gamma, T \rangle.$$

Since Equation (5) gives $0 < |\langle \Gamma, T \rangle| < 1$, Equation (6) and the bound on degrees $\delta \leq d^k$ then implies $0 < |\langle \Lambda, T \rangle| < 1$.

By definition $lk(\eta, T) := \langle \Lambda, T \rangle \pmod{1} \neq 0 \pmod{1}$. Since linking numbers are a well-defined invariant of the homology class, this gives that $[\eta] \neq 0$.

Hence, for any k we have shown that there is some lift η of γ under f^k with η representing a non-trivial homology class. Corollary 4.4 implies that $H_1(W^s([0 : 1 : 0]))$ is infinitely generated. □

7. FURTHER APPLICATIONS

In this final section we discuss a few examples of maps to which we have applied the results of this paper, and then a few types of maps which we feel would be fruitful to study further with techniques similar to those of this paper.

7.1. Relationship between connectivity of J_2 and the topology of the Fatou set for polynomial skew products. Recall that for polynomial skew products, $J_2 = \text{supp}(\mu) = \text{supp}(T \wedge T) = \overline{\bigcup_{z \in J_p} J_z}$, which by [19] is also the closure of the set of repelling periodic points. Here we examine to what extent connectivity of J_2 affects the homology of the Fatou set U .

The following example shows that there are many polynomial skew products f with J_2 connected for which $H_1(U(f))$ is non-trivial (in fact infinitely generated.)

Example 7.1. Consider $f(z, w) = (z^2 - 2, w^2 + 2(2 - z))$ which has J_2 connected and has J_z disconnected over $z = -2 \in J_p$, as shown in [19, Example 9.7]. Theorem 1.3 immediately applies, giving that $H_1(U(f))$ is infinitely generated.

In fact, examples of this phenomenon can appear “stably” within a one parameter family. Let $p_n(z) = z^2 + c_n$ be the unique quadratic polynomial with periodic critical point of least period n and c_n real. Then, [8, Theorem 6.1] yields that for n sufficiently large,

$$f_n(z, w) = (p_n(z), w^2 + 2(2 - z))$$

is Axiom A with J_z disconnected for most $z \in J_{p_n}$ and with J_2 connected. Suppose that f_n is embedded within any holomorphic one-parameter family $f_{n,\lambda}$ of polynomial skew products. Then, Theorems 4.1 and 4.2 from [8] (see also, [18, Thm C]) give that all maps $f_{n,\lambda}$ within the same hyperbolic component as f_n also have J_2 connected, but J_z disconnected over most z in $J_{p_{n,\lambda}}$. (Here, $p_{n,\lambda}$ is the first component of $f_{n,\lambda}$.) An immediate application of Theorem 1.3 yields that $H_1(U(f_{n,\lambda}))$ is infinitely generated for all $f_{n,\lambda}$ within this hyperbolic component.

Next we consider the possibility of J_2 being disconnected, but f not satisfying the hypotheses of our Theorem 1.3.

Question 7.2. Is there a polynomial skew product f with J_2 disconnected, but all J_z 's connected for all $z \in J_p$, such that $H_1(U(f))$ is trivial? More generally, is there any endomorphism of \mathbb{P}^2 with J_2 disconnected, but with all Fatou components having trivial homology?

By [19, Proposition 6.6], in order for a polynomial skew product f to satisfy the hypotheses of this question, J_p would have to be disconnected. However, a simple product like $(z, w) \mapsto (z^2 - 100, w^2)$ does not suffice; note for this map, the basin of attraction of $[1 : 0 : 0]$, hence the Fatou set, has nontrivial homology. Not many examples of non-product polynomial skew products are understood, and the current list of understood examples contains no maps which satisfy the hypotheses of this question.

7.2. A quadratic family of polynomial skew products. We now consider the family of examples $F_a(z, w) = (z^2, w^2 + az)$, which are skew products over $p(z) = z^2$.

The geometry and dynamics in $J_p \times \mathbb{C}$ were explored in [8]. For example, there it is established that:

- (1) [8, Theorem 5.1]: F_a is Axiom A if and only if g_a is hyperbolic; and
- (2) [8, Lemma 5.5]: J_2 can be described geometrically in the following manner: $J_{e^{it}}$ is a rotation of angle $t/2$ of $J_{\{z=1\}}$. That is, start with $J(g_a)$ in the fiber $J_{\{z=1\}}$, then as the base point $z = e^{it}$ moves around the unit circle $J_p = S^1$, the corresponding J_z 's are rotations of $J(g_a)$ of angle $t/2$, hence the J_z 's complete a half turn as z moves once around the base circle.

Due to the structure of J_2 , the difference between F_a and the product $H_a(z, w) = (z^2, w^2 + a)$ is one “twist” in J_2 . In [8] it is shown that F_a and H_a are in the same hyperbolic component if and only if a is in the main cardioid of the Mandelbrot set, \mathcal{M} .

Proposition 7.3. *Consider the family of mappings $F_a(z, w)$.*

- *If $a \in \mathcal{M}$, then the entire Fatou set of F_a has trivial homology.*
- *If $a \notin \mathcal{M}$, then for F_a , $W^s([0 : 1 : 0])$ has infinitely generated first homology.*

Proof. The second case follows directly from Theorem 6.4 after we observe that over the fixed fiber $z = 1$ we have $J_z = J(g_a)$, where $g_a(w) = w^2 + a$.

The first case can be proved using very similar techniques to those from Sections 4 and 7 of [24]. We outline the necessary modifications.

Let S_0 be the unit circle in the line $z = 0$ and S_Π be the unit circle in the line at infinity Π . Note S_0 and S_Π are saddle basic sets, with $S_\Pi = J_\Pi$ and $S_0 = J_{z=0}$. Similar to Section 4 of [24], one can form global separatrices $W^s(S_0)$ and $W^s(S_\Pi)$ that are locally defined as the vanishing set of a real-analytic function. One can also assume that these local defining functions are pluriharmonic. (See the details of the proof of Proposition 4.4 of [24].)

Using the dynamics of the base map $p(z) = z^2$ one can see that points in any Fatou component have orbits converging to either the line $z = 0$ or to the line at infinity Π . Coupled with the existence of the stable separatrices $W^s(S_0)$ and $W^s(S_\Pi)$ and the dynamics of F_a restricted to the lines $z = 0$ and Π , this implies that the Fatou set of F_a is the union of basins of attraction for three superattracting fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. A Mayer-Vietoris proof like the one in [24] shows that if we can prove that $W^s(S_0)$ and $W^s(S_\Pi)$ are homotopy equivalent to the unit circle, then each of the basins $W^s([1 : 0 : 0])$, $W^s([0 : 1 : 0])$, and $W^s([0 : 0 : 1])$ has trivial homology.

Using the symmetry $[Z : W : T] \mapsto [T : W : Z]$ for F_a , it suffices to study $W^s(S_0)$.

Note that for $|z| < 1$ we have that J_z is the intersection of $W^s(S_0)$ with the vertical line $\{z\} \times \mathbb{C}$ and that K_z is the intersection of $W^s([0 : 0 : 1]) \cup W^s(S_0)$ with $\{z\} \times \mathbb{C}$. For $|z|$ sufficiently small this follows from the existence of the stable manifold $W^s(S_0)$ and the dynamics in the line $z = 0$; then the fiber-wise invariance of K_z and J_z under F_a implies the result for all z with $|z| < 1$.

Consider the fiber-wise critical points $C_z := \{w \in \mathbb{C} : q'_z(w) = 0\}$. If $a \in \mathcal{M}$ we have that J_z is connected over the fixed fiber $\{z = 1\}$. Therefore, Theorem 5.5 from [8] implies that J_z is connected for every $z \in J_p$ and so that Theorem 6.4 from [19] gives that $C_z \subset K_z$ for all $z \in \mathbb{C}$.

The union of these fiber-wise critical points is just the horizontal line $w = 0$ and the results from the previous paragraph show that this line stays on one side of $W^s(S_0)$, possibly touching at many points. Note, however that they are disjoint at $z = 0$.

Consider the point z_0 (with $|z_0| < 1$) of smallest modulus where $w = 0$ and $W^s(S_0)$ touch. Then, there is a neighborhood of U of $(z_0, 0)$ in \mathbb{C}^2 in which $W^s(S_0)$ is given by the zero set of a pluriharmonic function Ψ . Changing the sign of Ψ (if necessary) we can assume that $\Psi \leq 0$ for points in $K_z \cap U$. The restriction $\psi(z) = \Psi|_{w=0}$ is a non-positive harmonic function in a neighborhood of z_0 having $\psi(z_0) = 0$, but $\psi(z) < 0$ for z with $|z| < |z_0|$. This violates the maximum principal.

Therefore, $W^s(S_0)$ is disjoint from the critical line $w = 0$. Then a Morse-theoretic proof nearly identical to the one in Section 7 of [24] can be used to show that $W^s(S_0)$ is homotopy equivalent to S_0 . \square

7.3. Postcritically Finite Holomorphic Endomorphisms. Until presenting the conjecture of the previous subsection, this paper has been about endomorphisms with complicated Fatou topology. The opposite extreme is that the Fatou topology may also be trivial in many cases. We suspect one simple case in which Fatou topology is trivial is when the map is postcritically finite (PCF).

Question 7.4. Does the Fatou set of a postcritically finite holomorphic endomorphism of \mathbb{P}^2 always have trivial homology?

A starting point for investigation into this question could be to attempt to establish it for the postcritically finite examples constructed by Sarah Koch [20, 21]. Heuristic evidence supports that the homology is trivial for Koch's maps. Her construction provides a class of PCF endomorphisms, containing an infinite number of maps, including the previously studied examples of [10] and [7].

7.4. Other holomorphic endomorphisms of \mathbb{P}^2 . As we have demonstrated in Sections 5 and 6.3, given some information about the geometry of the support of T , we can apply the techniques of Sections 3 and 4 to study the Fatou set of a holomorphic endomorphism of \mathbb{P}^2 . We would like to be able to apply this theorem to other holomorphic endomorphisms. However, specific examples of holomorphic endomorphisms that are amenable to analytic study are notoriously difficult to generate.

One family of endomorphisms which seem a potentially vast area of study are the Hénon-like endomorphisms. Introduced by Hubbard and Papadapol in [16], and studied a bit further by Fornæss and Sibony in [11], these are holomorphic endomorphisms arising from a certain perturbation of the Hénon diffeomorphisms. The Hénon diffeomorphisms have been deeply studied (e.g., by Bedford Lyubich, and Smillie, [1, 2], Bedford and Smillie [3, 4], Hubbard and Oberste-Vorth [14, 15], and Fornæss and Sibony [9]). A natural question which is thus far quite wide open is: how does the dynamics of a Hénon diffeomorphism relate to the dynamics of the perturbed Hénon endomorphism? Computer evidence suggests the dynamics of Hénon-like endomorphisms is rich and varied.

Specifically concerning the topology of the Fatou set, the main result of [3] is that connectivity of the Julia set is determined by connectivity of a slice Julia set in a certain

unstable manifold. We ask whether this result would have implications for the related Hénon endomorphism, which would allow us to use Theorem 1.1 to establish some analog of Theorem 1.3 for Hénon endomorphisms.

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