MAT545: Assignment V

Due Tuesday, May first 2007

1. Let $L \to M$ be a Hermitian line bundle (holomorphic with a Hermitian metric) over a complex manifold M and let ∇ denote its Chern connection. Suppose that $e : \mathcal{U} \to L$ is a local non-vanishing holomorphic section of L where $\mathcal{U} \subset M$ is some open set. This induces a local trivialization

$$\begin{array}{rccc} \phi : & \mathcal{U} \times \mathbb{C} & \to & L|_{\mathcal{U}} \\ & (u, \lambda) & \mapsto & \lambda e(u). \end{array}$$

(a) In this local trivialization, the Chern connection takes the form $\nabla = d + \theta$ in the sense that

$$\nabla \lambda e = d\lambda e + \theta \lambda e$$

where θ is the connection one form. Show that θ is explicitly given by:

$$\theta = \partial \log |e|^2.$$

Why is it a $(1,0)$-form?

(b) Conclude that in the open set $\mathcal{U}$, the curvature form is given by

$$\Theta = \overline{\partial}\partial \log |e|^2.$$

(c) Consider the particular case where $L \to \mathbb{P}^n$ is the tautological line bundle over $\mathbb{P}^n$. Since each fibre of L naturally sits in $\mathbb{C}^{n+1}$, the hermitian metric on $\mathbb{C}^{n+1}$ naturally induces one on L. Thus L can be seen as a Hermitian line bundle. Show that the Fubini-Study metric is given by

$$\omega_{FS} = -\frac{\sqrt{-1}}{2\pi}\Theta_L$$

where Θ_L is the curvature form associated to the corresponding Chern connection.

Remark 1 *The factor of* $\frac{\sqrt{-1}}{2\pi}$ *is there to insure that the cohomology class* $[\omega_{FS}]$ *is integral.*

(d) What is the sign of the curvature of the tautological line bundle? Is it consistent with the fact that ω_{FS} is a Kähler form?

(e) Does the tautological line bundle $L \to \mathbb{P}^n$ admit a global holomorphic section? (Hint: You can use the fact that $[\omega_{FS}] = -c_1(L)$ is minus the first Chern class of L).

2. Let g be the Kähler metric of a Kähler manifold M. Show that if $g' := e^f g$ is another Hermitian metric conformal to g where $f \in \mathcal{C}^\infty(M)$ is a smooth real valued function, then g' is Kähler if and only if f is constant.

3. Consider the group $\Gamma := \mathbb{Z}^2 \times \mathbb{Z}^2$ with noncommutative group operation

$$(j', k') \circ (j, k) = (j + j', A_{j'} k + k'), \quad A_j := \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix}$$

where $j = (j_1, j_2) \in \mathbb{Z}^2$ and similarly for k. This group acts on $\mathbb{R}^4$ via

$$\begin{array}{cccc} \rho : & \Gamma & \to & \mathrm{Diff}(\mathbb{R}^4) \\ & (j,k) & \mapsto & \rho_{jk} \end{array}$$

where $\rho_{jk}(x, y) = (x + j, A_j y + k)$.

(a) Show that the diffeomorphisms ρ_{jk} preserve the standard symplectic form

$$\omega := dx_1 \wedge dx_2 + dy_1 \wedge dy_2.$$

Conclude that this symlectic form naturally descends to give the quotient $M := \mathbb{R}^4/\Gamma$ a structure of compact symplectic manifold.

(b) Show that the fundamental group of M is $\pi_1(M) \cong \Gamma$.

(c) Compute the first homology group of M via the relation

$$H_1(M) = \pi_1(M)/[\pi_1(M), \pi_1(M)].$$

(d) Conclude via the Hodge identities that the compact symplectic manifold M cannot be given a structure of Kähler manifold. This manifold, found by Thurston in the 1970s, was the first example of a symplectic manifold which is not Kähler.

4. Let ω be the Kähler form of a compact Kähler manifold. Show that ω is a harmonic $(1,1)$-form.