MAT545: Assignment IV

Due Thursday, March 29th 2007

1. Show that the (oriented) intersection number of two distinct compact complex submanifolds of a compact complex manifold is always non-negative. Can you find an example where it is zero?

2. Compute the Dolbeault cohomology of $\mathbb{P}^2$ using an acyclic cover.

3. Let H be the hyperplane bundle over the projective space $\mathbb{P}^n$. It is the dual of the tautological line bundle. In particular, it is a holomorphic line bundle. Let $H^{\otimes d}$ be the tensor product of d copies of H on $\mathbb{P}^n$. The zero locus of a holomorphic section of $H^{\otimes d}$ is a hypersurface of degree d in $\mathbb{P}^n$. If the section is transversal to the zero section, then the hypersurface is a smooth complex submanifold.

(a) In problem 3. of Assignment III, the space of all hypersurfaces of degree d was described as a projective space. Using Sard's theorem and problem 3.(b) of Assignment II, show that almost all (in the sense of measure theory) hypersurfaces of degree d are smooth.

(b) In problem 3(c) of Assignment II, you were supposed to find out that the space $Sing \subset \tilde{V}$ was an analytic subvariety of complex codimension n in $\tilde{V}$ where $n := \dim_{\mathbb{C}} \mathbb{P}(S) = \dim_{\mathbb{C}} S - 1$. Invoking the proper mapping theorem (see p.34 in Griffith and Harris), conclude from this that the generic (in the sense of algebraic geometry) hypersurface of degree d is smooth.

(c) Compute the (oriented) intersection number between a smooth hypersurface of degree d in $\mathbb{P}^n$ and a line $\mathbb{P}^1 \subset \mathbb{P}^n$.

(d)[1] Compute the Chern classes of a smooth hypersurface of degree d in $\mathbb{P}^n$. When is the first Chern class zero? When this happens, this gives an example of a Calibi-Yau manifold, which is a complex manifold admitting a Ricci flat Kähler metric.

(e) Compute the self-intersection of a smooth hypersurface of degree d in $\mathbb{P}^2$. Can this self-intersection be negative? Would that contradict the statement of problem 1.?

(f) Let $[x_0 : x_1 : x_2]$ be the standard homogeneous coordinates for $\mathbb{P}^2$. Show that the hypersurface of degree 2 given by

$$\Sigma := \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \quad | \quad x_0^2 + x_1^2 + x_2^2 = 0\} \subset \mathbb{P}^2$$

is smooth. Compute its intersection number with a (smooth) hypersurface of degree $d > 0$.

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[1]cf. exercise 21.14, p.282 in the book of Bott and Tu *Differential forms in algebraic topology*