MAT545: Assignment II

Due Tuesday, February 20th 2007

1. Let Ω be an open connected set in $\mathbb{C}^n$ so that there is a smooth proper function $\phi : \mathbb{C}^n \to [-1, \infty)$ such that:

(i) 0 is a regular value;

(ii) $\phi^{-1}((0, \infty))$ is connected.

(iii) $\Omega = \phi^{-1}([-1, 0))$ is connected.

Suppose further that u is a $\mathcal{C}^\infty$ function on $\partial\Omega$. Let $\tilde{u}$ be an extension of u to a neighborhood of Ω.

(a) Show that $\overline{\partial}\phi$ is nowhere zero on $\partial\Omega$.

(b) Show that the condition

$$\overline{\partial}\phi \wedge \overline{\partial}\tilde{u}\big|_{\partial\Omega} = 0$$

is independent of the choice of extension of u and the choice of ϕ.

(c) For the function u satisfying the condition in (b), Show that there is an extension U_1 to $\overline{\Omega}$ such that $U_1|_{\partial\Omega} = u$ and $\frac{\overline{\partial}U_1}{\phi^2}$ is a uniformly bounded one-form on Ω.

(d) For the function u satisfying the condition in (b), show that if $n > 1$, there is a holomorphic extension $U : \overline{\Omega} \to \mathbb{C}$ such that

$$U|_{\partial\Omega} = u.$$

(**Hint:** Prove that if $f \in \mathcal{C}^\infty(\mathbb{C}^n)$ is a smooth function such that $f(\partial\Omega) = 0$ then there exists a smooth function g such that $f = \phi g$ in a neighborhood of $\partial\Omega$, this should help in (b) and (c))

2. Let $g : \mathbb{C} \to \mathbb{C}$ be a compactly supported L^p-function. If

$$f := \frac{1}{2\pi i} \int_{\mathbb{C}} g(u) \frac{d\overline{u} \wedge du}{u - z},$$

then for any ball $B_r(0)$ of radius r centered at the origin, there is a constant C only depending on p, r and the support of g so that

$$\|f\|_{L^p(B_r(0))} \leq C\|g\|_{L^p(\mathbb{C})}.$$

(**Hint:** Hölder inequality...)

3. Let S be a (complex) vector space. Let $\mathbb{P}(S)$ denote its projective space and let $H^d = H^d(S)$ denote the space of homogeneous polynomials of degree d and H^d_o the subspace of non-zero polynomials. Let ev be the evaluation map

$$ev : S \times H^d_0 \to \mathbb{C}.$$

Given a polynomial $p \in H^d_o$, let $V_p \subset \mathbb{P}(S)$ denote the projectivization of the zero set of p. Prove the following

(a) V_p is smooth if for all $s \in S$, we have $d_s p \neq 0$.

(b) Let $\tilde{V} \subset \mathbb{P}(S) \times \mathbb{P}(H^d)$ denote the universal zero set

$$\tilde{V} := \{[s], [p]) \quad | \quad p(s) = 0\}.$$

From its definition $\tilde{V}$ admits a pair of projections $\pi_1 : \tilde{V} \to \mathbb{P}(S)$ and $\pi_2 : \tilde{V} \to \mathbb{P}(H_d)$. Show that $\tilde{V}$ is smooth and that V_p is smooth at $[s]$ if the differential of $\pi_2 : \tilde{V} \to \mathbb{P}(H^d)$ at $([s], [p])$ is surjective.

(c) Let $Sing \subset \tilde{V}$ be the set of $([s], [p])$ where the differential of π_2 is not surjective. What is the dimension of $Sing$?