Math 122 (Fall '12)

Review Questions for Final (Partial Solutions)

Disclaimer: You should use these solutions only **after** you tried to solve the exercises yourself. Since I prepared these solutions in a hurry, there might be typos and small computational mistakes. Please make sure that you understand the arguments, and don't focus exclusively on the answer.

Part I - Fundamental Questions

1. Solve the following equations:

i. Linear Equations: $2(x+3) = 5$

Answer: $x = -\frac{1}{2}$

ii. Quadratic Equations: $2x^2 + 5x = 3$

Answer: Factor as $(2x-1)(x+3) = 0$. Thus the *two solutions* are $x = \frac{1}{2}$ and $x = -3$.

Alternatively, you could apply the general formula for quadratic equations of type

$$ax^2 + bx + c = 0.$$

Namely, the solutions are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ for $\Delta = b^2 - 4ac$. Here $\Delta = 49$. Thus $x_{1,2} = \frac{-5 \pm 7}{4}$, giving the solutions -3 and $\frac{1}{2}$ as before.

iii. Equations involving exponential functions: $2 \cdot e^x = 3 \cdot 2^x$

Answer: Take ln of both sides. Get

$$\ln 2 + x = \ln 3 + x \ln 2,$$

Thus,

$$\begin{aligned} x(1-\ln 2) &= \ln 3 - \ln 2 \\ x &= \frac{\ln 3 - \ln 2}{1 - \ln 2} \end{aligned}$$

iv. Equations involving logarithms: $2\ln x = \ln(3x) + 5$

Answer: Get

$$2\ln x = \ln x + \ln 3 + 5$$

Thus

$$\ln x = \ln 3 + 5 = \ln 3 + \ln e^5 = \ln 3e^5$$

Exponentiate both sides and get the final answer

$$x = 3e^5.$$

v. Equations involving power functions: $x^{100} = 2^{99} \cdot \sqrt{x^{101}}$

Answer: Write the equation in power form

$$x^{100} = 2^9 9 \cdot x^{\frac{101}{2}}$$

Thus

$$\begin{aligned} x^{100-\frac{101}{2}} &= 2^{99} \\ x^{\frac{99}{2}} &= 2^{99} \end{aligned}$$

Raising both sides to power $\frac{2}{99}$ gives the final answer

$$x = 2^2 = 4$$

2.

i. $3x^2 + 2e^x + \sqrt{x^3} + \frac{4}{x} - \ln x$ (basic rules)

Answer:

$$6x + 2e^x + \frac{3}{2}\sqrt{x} - \frac{4}{x^2} - \frac{1}{x}$$

ii. $(x^2 + 1)\ln x$ (product rule)

Answer:

$$2x \ln x + \frac{x^2 + 1}{x}$$

iii. $\ln(x^2 + 1)$ (chain rule)

Answer:

$$\frac{2x}{x^2 + 1}$$

iv. $\frac{x}{x^2+1}$ (quotient rule)

Answer:

$$\frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

v. $e^{x \ln x} + xe^{2 \ln x + 3}$ (both chain and product rules)

Answer:

$$e^{x \ln x}(\ln x + 1) + e^{2 \ln x + 3} + xe^{2 \ln x + 3} \cdot \frac{2}{x}$$

3. Find an antiderivative for each of the following functions

i. $3x^2 + 2e^x + \sqrt{x^3} + \frac{4}{x}$ (basic rules)

Answer:

$$x^3 + 2e^x + \frac{2}{5}x^{\frac{5}{2}} + 4\ln x + C$$

ii. $\frac{\sqrt{\ln x}}{x}$ (substitution) **Answer:**

$$\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}$$

Thus

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$$

iii. $x^2 e^{x^3+1}$ (substitution)

Answer:

$$\begin{aligned} u &= x^3 + 1 \\ du &= 3x^2 dx \\ \frac{du}{2} &= x^2 dx \end{aligned}$$

Thus,

$$\int x^2 e^{x^3+1}\,dx = \int e^u \frac{du}{3} = \frac{e^u}{3} = \frac{e^{x^3+1}}{3} + C$$

iv. $x(2x^2+3)^{100}$ (substitution)

Answer:

$$\begin{aligned} u &= 2x^2 + 3 \\ du &= 4x dx \end{aligned}$$

Get

$$\int u^100 \frac{du}{4} = \frac{u^{101}}{4 \cdot 101} = \frac{(2x^2+3)^{101}}{404} + C$$

v. $x^2(x^5 - x^6)$ (expand, basic rules)

Answer:

$$\int x^2(x^5 - x^6)\ dx = \int x^7 - x^8\ dx = \frac{x^8}{8} - \frac{x^9}{9} + C$$

4. Questions related to the fundamental theorem of calculus

A. Compute the definite integrals

i) $\int_0^1 2x - e^x\ dx$

Answer:

$$\int_0^1 2x - e^x\ dx = (x^2 - e^x)\ |_0^1 = (1 - e^1) - (0 - e^0) = 2 - e$$

ii) $\int_1^4 x\sqrt{x^2+1}\ dx$

Answer: We compute first the antiderivative of $x\sqrt{x^2+1}$ by using the substitution

$$\begin{aligned} u &= x^2 + 1 \\ du &= 2xdx \end{aligned}$$

Thus

$$\int x\sqrt{x^2+1}\ dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}(x^2+1)^{\frac{3}{2}}$$

Now, go back to the original question and get

$$\int_1^4 x\sqrt{x^2+1}\ dx = \frac{1}{3}(x^2+1)^{\frac{3}{2}}\ |_1^4 = \frac{1}{3}\left(17^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

B. Is it true that

i) $\int xe^x \, dx = (x-1)e^x + C$?

Answer: Compute the derivatives of both sides. By the Fundamental Theorem of Calculus, on the LHS we get the original function, i.e. xe^x, on the RHS we need to compute the derivative of $(x-1)e^x$. In other words, the original question is equivalent to the equality

$$xe^x = (x-1)e^x)'???$$

Now

$$(x-1)e^x)' = e^x + (x-1)e^x = xe^x$$

So indeed the above equality holds, so the final answer is TRUE.

ii) $\int x \ln x \, dx = (x-1)\ln x + C$? **Answer:**

$$x \ln x = ((x-1)\ln x)'???$$

But now,

$$((x-1)\ln x)' = \ln x + \frac{x-1}{x}$$

Thus, the final answer is FALSE.

Part II - Applications of derivatives/integrals

7. (Integrals) Estimation of the integral using Riemann Sums (see [SFinal](8)]. For instance:

i) Estimate $\int_0^2 x^2 dx$ using 4 division points and using

   Answer: We divide the interval in 4 pieces. Thus the division points are 0, 0.5, 1, 1.5 and 2. The size of the small intervals is $\Delta x = 0.5$.

   – the left end-point

     Answer:

$$f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x =$$

$$= 0^2 \cdot 0.5 + 0.5^2 \cdot 0.5 + 1^2 \cdot 0.5 + 1.5^2 \cdot 0.5 = 1.75$$

   – the right end-point

     Answer:

$$f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x = 3.75$$

   – the mid-point

     Answer:

$$f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x = 2.625$$

   Which is one is the most accurate method? Which one is clearly an underestimate? Which one is clearly an overestimate? Sketch a graph

   Answer: Underestimate: Left. Overestimate: Right. Best Estimate: Midpoint.

ii) What is a reasonable values for $\int_1^3 f(x)\ dx$ if you are given

x	0	0.5	1	1.5	2	2.5	3
f(x)	1	1.2	1.5	2	1.4	1.1	0.7

? **Answer:** Here you could use either left or right Riemann sums to estimate the integral (either one of them is acceptable for the final). We get

$$\int_1^3 f(x)\ dx \sim f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x =$$
$$= 1.5 \cdot 0.5 + 2 \cdot 0.5 + 1.4 \cdot 0.5 + 1.1 \cdot 0.5 = 3$$

8. (Integrals) Compute areas (see [SFinal](7)). For instance

i) Find the area below the graph for $\sqrt{x}$ for $x \in [1, 9]$. **Answer:**

$$\text{Area} = \int_1^9 \sqrt{x}\ dx = \frac{2}{3}x^{\frac{3}{2}}\ |_1^9 = \frac{2}{3}(9^{\frac{3}{2}} - 1) = \frac{52}{3}$$

Note $x^{\frac{3}{2}} = x\sqrt{x}$. Thus $9^{\frac{3}{2}} = 9\sqrt{9} = 27$.

ii) Find the area between the graphs of the functions $f(x) = 3x - 2$ and $g(x) = x^2$. (**Note:** the original equations do not work; there is a single point of intersection.)

Answer: Note that you are not given the interval over which to integrate. Thus, you need to find the intersection points for the 2 curves (draw a graph). In other words, you need to solve

$$3x - 2 = x^2$$

One gets, $x = 1$ and $x = 2$. Thus, we need to compute

$$\text{Area} = \int_1^2 (3x-2) - x^2\ dx = \frac{3}{2}x^2 - 2x - \frac{1}{3}x^3\ |_1^2 = (\frac{12}{2} - 4 - \frac{8}{3}) - (\frac{3}{2} - 2 - \frac{1}{3})$$

Thus,

$$\text{Area} = 4 - \frac{3}{2} - \frac{7}{3} = \frac{1}{6}$$

(**Again, I warn you that the numerics might be wrong. Please check the answers carefully.**)