Maximal Cohen-Macaulay modules over $Y_1^3 + \cdots + Y_n^3$ with few generators

Radu Laza[1], Liam O'Carroll[2] and Dorin Popescu[3*]

Dedicated to Professor Nicolae Radu on his 70^{th} birthday

Introduction

Let K be an algebraically closed field with char $K \neq 3$, $\{Y_1, \ldots, Y_n\}$ a set of indeterminates over K, and let $R_n = K[Y_1, \cdots, Y_n]/(f_n)$, where $f_n = Y_1^3 + \ldots + Y_n^3$. In this paper we consider graded maximal Cohen-Macaulay modules over R_n that have no free direct summands; for simplicity, we call these *standard maximal Cohen-Macaulay* (MCM) *modules* (over R_n). In particular, we are interested in the question of the existence of such modules with bounded number of generators. For example we prove the following result in Theorem 3.1 :

Theorem 0.1 *For a positive integer q, if $n > 2q$, then there exist no standard maximal Cohen-Macaulay modules that are generated by q elements.*

Because of this result, we focus on the the function $\mu(R_n)$ of n, defined to be the minimum of the number of generators of standard MCMs over R_n, especially for small values of n. Indeed we are interested in obtaining nice families of standard MCMs, especially for such values of n. Of course our approach is based mainly on Eisenbud's theory of (reduced) matrix factorizations [Eis], but we also use substantially his insight into the rôle played by determinants [Eis, (6.4)]. We were also inspired by Yoshino's theory of tensor products of matrix factorizations [Yos1]. Our main results in this direction give the following information (see 1.1, 3.2, 3.3, 3.5, 3.6 and 3.8).

Theorem 0.2 *(i) $\mu(R_2) = 1$, $\mu(R_3) = 2$, $\mu(R_4) = 2$, $\mu(R_5) = 4$, $\mu(R_n) \geq \mu(R_{n-1})$.*
(ii) $\mu(R_n) \geq n/2$ for $n \geq 1$.
(iii) Suppose $n \geq 3$ and $\mu(R_{n-1}) < \mu(R_n)$. Then $2\mu(R_{n-1}) \geq \mu(R_n)$.

*The third author's research was supported by a grant from EPSRC.This support is gratefully acknowledged.

A result of Bruns, given in [He-Kü, (3.4)] proved very useful in giving a more complete version of parts of our work (see 3.3(b) and 3.8 below). We would like to thank J. Herzog for bringing this paper to our attention. It uses a wide variety of sophisticated techniques to obtain interesting results on the rank and minimal number of generators of maximal Cohen-Macaulay modules, but in a different direction to our investigations.

While we have to content ourselves with presenting nice families of standard MCMs with few generators in case $n = 3$ and 4 (see 2.1, 2.2, 3.8 and 3.9), in the case $n = 3$ we can classify completely isomorphism classes of standard MCMs with two generators in a particularly satisfying way (see 1.1).

Proposition 0.3 *There is a bijection between the isomorphism classes of standard MCM modules over R_3 with two generators and points of the projective plane cubic $X_0^3 + X_1^3 + X_2^3 = 0$.*

This results calls to mind the very general work of Kahn [Ka], but the bijection in our situation is quite simple and direct.

For general background, see [Br-He] and [Yos2].

1 Two-generated standard maximal Cohen-Macaulay modules over R_3

To describe such modules is equivalent, by the work of Eisenbud [Eis], to describing the reduced 2×2-matrix factorizations of f_3 over $K[Y_1, Y_2, Y_3]$ with homogeneous entries. That is, we wish to find 2×2-matrices ϕ and ψ, with entries homogeneous forms of positive degree from $K[Y_1, Y_2, Y_3]$, such that $\phi\cdot\psi = f_3\cdot I_2$, I_2 being the 2×2 identity matrix. Then $\det\phi \cdot \det\psi = f_3^2$ and since f_3 is irreducible we see that $\det\phi = \det\psi = f_3$. In fact we see that $\psi = \mathrm{adj}(\phi)$, the adjoint matrix of ϕ, so that it suffices to find such ϕ with $\det\phi = f_3$.

Let $\lambda = [\lambda_1 : \lambda_2 : \lambda_3]$ be a point on the curve $X \subseteq \mathbf{P}^2_K$ with equation $f_3 = 0$. After a suitable change of coordinates if necessary,we may suppose that $\lambda_3 \neq 0$ and so may take $\lambda_3 = 1$. Thus $\lambda_1^3 + \lambda_2^3 + 1 = 0$. Then $f_3 = (Y_1^3 - \lambda_1^3 Y_3^3) + (Y_2^3 - \lambda_2^3 Y_3^3) = (Y_1 - \lambda_1 Y_3)(Y_1^2 + \lambda_1 Y_1 Y_3 + \lambda_1^2 Y_3^2) + (Y_2 - \lambda_2 Y_3)(Y_2^2 + \lambda_2 Y_2 Y_3 + \lambda_2^2 Y_3^2)$, that is, $f_3 = \det\phi_\lambda$ where

$$\phi_\lambda = \begin{pmatrix} Y_1 - \lambda_1 Y_3 & Y_2 - \lambda_2 Y_3 \\ -(Y_2^2 + \lambda_2 Y_2 Y_3 + \lambda_2^2 Y_3^2) & Y_1^2 + \lambda_1 Y_1 Y_3 + \lambda_1^2 Y_3^2 \end{pmatrix}.$$

We can now give the main result of this section. Although it is reminiscent of Kahn's general correspondence [Ka], the correspondence it presents is particularly immediate and direct.

Proposition 1.1 *The correspondence $\Phi : \lambda \rightarrow \phi_\lambda$ defines a bijection between the points of the curve X and the isomorphism classes of standard maximal Cohen-Macaulay modules over R_3 having two generators.*

Proof: Suppose that λ, λ' are two distinct points on X. After a suitable change of coordinates if necessary, we may suppose that $\lambda_3 = \lambda'_3 = 1$ (in an obvious notation). Then the Fitting ideal $\mathrm{Fit}_1(\phi_\lambda)$ is the ideal $(Y_1-\lambda_1 Y_3, Y_2-\lambda_2 Y_3, Y_1^2+\lambda_1 Y_1 Y_3+\lambda_1^2 Y_3^2, Y_2^2+\lambda_2 Y_2 Y_3+\lambda_2^2 Y_3^2)$, i.e. $\mathrm{Fit}_1(\phi_\lambda) = (Y_1-\lambda_1 Y_3, Y_2-\lambda_2 Y_3, Y_3^2)$. It follows almost immediately that $\mathrm{Fit}_1(\phi_\lambda) \neq \mathrm{Fit}_1(\phi_{\lambda'})$. Thus Φ is injective.
Now let A be a 2×2 matrix with entries homogeneous forms in $K[Y_1, Y_2, Y_3]$ of positive degree such that $\det A = f_3$; then A and its adjoint matrix $\mathrm{adj}(A)$ form a matrix factorization of a standard maximal Cohen-Macaulay module M over R_3 having two generators. After elementary transformations on A (which correspond to isomorphisms on M) we may suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \deg a = \deg b = 1, \deg c = \deg d = 2,$$

with $a = Y_1 - \mu_1 Y_3$, $b = Y_2 - \mu_2 Y_3$ for $\mu_1, \mu_2 \in K$. Since $\det A = f_3$, $f_3(\mu_1, \mu_2, 1) = 0$; that is, $\mu_1^3 + \mu_2^3 + 1 = 0$ and therefore $\mu = [\mu_1 : \mu_2 : 1]$ is a point on X. For the surjectivity of Φ, it is enough to show that A and ϕ_μ are equivalent, that is, they define the same maximal Cohen-Macaulay module M. Set $c' = -(Y_2^2 + \mu_2 Y_2 Y_3 + \mu_2^2 Y_3^2)$ and $d' = Y_1^2 + \mu_1 Y_1 Y_3 + \mu_1^2 Y_3^2$. Then

$$ad' - bc' = f_3 = ad - bc,$$

so $a(d-d') = b(c-c')$. By unique factorization in $K[Y_1, Y_2, Y_3]$, we deduce that $d - d' = bl$, $c - c' = al$ for some linear form $l \in K[Y_1, Y_2, Y_3]$. Thus ϕ_μ arises from A by an elementary row operation, and so they are equivalent. ■

See also [OP, 7.7], where the decomposition

$$Y_1^3 + Y_2^3 + Y_3^2 = (Y_1 + Y_2 + Y_3)h + 3Y_1 Y_2 Y_3,$$

with $h = Y_1^2 + Y_2^2 + Y_3^2 - Y_1 Y_2 - Y_1 Y_3 - Y_2 Y_3$, gives a graded reduced matrix factorization (ϕ', ψ') over R_3 of size 2 such that $\det \phi' = f_3$.

2 Three-generated standard maximal Cohen-Macaulay modules over R_3

As in the first section, we see that a three-generated standard maximal Cohen-Macaulay module M over R_3 corresponds to 3×3 matrix ϕ of homogeneous forms of positive degree in $K[Y_1, Y_2, Y_3]$ such that $\det \phi$ equals f_3 or f_3^2; the matrix factorization is given by (ϕ,adj(ϕ)) in the first case (see [Eis, Section 5 and 6] for a general discussion). If necessary, replacing M by its first syzygy module, we may suppose that $\det\phi = f_3$. Since ϕ is 3×3, we deduce that the entries ϕ_{ij} of ϕ are linear forms. If one row or column of ϕ consists of linearly independent linear forms, then it is easily seen that after elementary row and column operations we may write

$$\phi = \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_2 & a & b \\ vY_3 & c & d \end{pmatrix}, \quad v \in K\setminus\{0\},$$

for some linear forms a, b, c, d.

Since $\det\phi = f_3$ it is easy to see that $a = a_1Y_1 + a_2Y_2 - v^{-1}Y_3$, $b = b_1Y_1 + b_2Y_2 + b_3Y_3$, $c = c_1Y_1 + c_2Y_2 + c_3Y_3$ and $d = d_1Y_1 - Y_2 + d_3Y_3$, where the $a_i, b_i, c_i, d_i \in K$ and satisfy the following identities:
(i) $a_1d_1 - b_1c_1 = 1$; (ii) $-a_1 + d_1a_2 - b_1c_2 - b_2c_1 = 0$;
(iii) $a_1d_3 - v^{-1}d_1 - b_1c_3 - b_3c_1 = 0$; (iv) $-d_3 + c_2 + vb_2 = 0$;
(v) $-d_1 - b_2c_2 - a_2 = 0$; (vi) $-va_2 + c_3 + vb_3 = 0$;
(vii) $-va_1 - b_3c_3 - v^{-1}d_3 = 0$; (viii) $v^{-1} + a_2d_3 + c_1 + vb_1 - b_2c_3 - b_3c_2 = 0$.
(These follow on equating coefficients of Y_1^3, $Y_1^2Y_2$, $Y_1^2Y_3$, $Y_2^2Y_3$, $Y_1Y_2^2$, $Y_2Y_3^2$, $Y_1Y_3^2$ and $Y_1Y_2Y_3$, respectively.)

From (iv) we have $d_3 = c_2 + vb_2$, from (vi) we have $a_2 = v^{-1}c_3 + b_3$, from (vii) $a_1 = -v^{-2}d_3 - v^{-1}b_3c_3$, from (viii) $c_1 = b_2c_3 + b_3c_2 - v^{-1} - a_2d_3 - vb_1$, and from (v) $d_1 = -a_2 - b_2c_2$. On substituting into (i), (ii), (iii) we get three equations $h_i(v, c_2, c_3, b_1, b_2, b_3) = 0$, $i = 1, 2, 3$, in six variables (the details are left to the reader). Hence we have:

Proposition 2.1 *Each point $(v, c_2, c_3, b_1, b_2, b_3)$, $v \neq 0$, of $V(h_1, h_2, h_3) \subseteq \mathbf{A}^6_K$ yields a 3-generated standard maximal Cohen-Macaulay module over R_3.*

Remark 2.2 *(a) A nice instance of such matrices is given by the family*

$$\phi_{uv} = \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_2 & -v^{-1}Y_3 & -uY_1 \\ vY_3 & u^{-1}Y_1 & -Y_2 \end{pmatrix}$$

where $u, v \in K\setminus\{0\}$ satisfy $-uv + u^{-1} + v^{-1} = 0$.
(b) It would seem to be a messy task to characterize isomorphism classes of the MCM modules given by 2.1, in contrast to the more satisfactory situation presented in 1.1.
(c) It is possible for ϕ to have dependence in a row or column (and then in each row and column). For example, for ϕ given as follows:

$$\phi = \begin{pmatrix} 0 & -(Y_1+Y_2) & Y_3 \\ Y_3 & 0 & Y_1 + \omega Y_2 \\ Y_1 + \omega^2 Y_2 & Y_3 & 0 \end{pmatrix},$$

where $\omega \in K$ satisfies $\omega^3 = 1$, we have $\det \phi = f_3$.
We could present a variant of 2.1 for ϕ of this sort (details are left to the interested reader).

We now recall a basic result of Eisenbud (see [Eis, (6.4)]), which we specialize to our situation.

Lemma 2.3 *A standard maximal Cohen-Macaulay module over R_3 given by a graded reduced $q \times q$ matrix factorization (ϕ, ψ) has rank k (and requires exactly q generators) if and only if* $\det \phi = f_3^k$ *and* $\mathrm{Fit}_1(\phi) \subseteq (f_3^{k-1})$.

So we quickly deduce the following result.

Proposition 2.4 *A 3-generated standard maximal Cohen-Macaulay module M over R_3 given by a graded reduced 3×3 matrix factorization (ϕ, ψ) has rank one if* $\det(\phi) = f_3$ *and rank two if* $\det(\phi) = f_3^2$.

Proof: If $\det \phi = f_3^2$, then $\det \psi = f_3$, so by 2.3 $\Omega^1_{R_3}(M)$ has rank 1. The short exact sequence

$$0 \to \Omega^1_{R_3}(M) \to R_3^3 \to M \to 0$$

then forces M to have rank 2.
If $\det \phi = f_3$, it follows immediately from 2.3 that rank $M = 1$ ■

So, in particular, the MCM modules given in 2.1 and 2.2 have rank one and so are Ulrich modules in the sense of [He-Kü].

3 Minimal number of generators of maximal Cohen-Macaulay modules over R_n

As before, let $R_n = K[Y_1, \ldots, Y_n]/(f_n)$, where $f_n = \sum_{i=1}^n Y_i^3$, and let q be a positive integer.

Theorem 3.1 *If $n > 2q$, then there does not exist a standard maximal Cohen-Macaulay module over R_n such that is generated by q elements.*

Proof: Suppose that there is such a module M, given by a reduced graded matrix factorization (ϕ, ψ) of size q over $K[Y_1, \ldots, Y_n]$; note that we may assume that q is the minimal number of generators, since decreasing q preserves the hypothesis and conclusion. Suppose that $\phi = (a_{ij})$, $\psi = (b_{jl})$. Then we have $f_n = \sum_{j=1}^q a_{ij} b_{ji}$ for each i, $1 \leq i \leq q$. Since $\deg a_{ij}$, $\deg b_{ji} \geq 1$ and $\deg f_n = 3$, we see that for each i,j, one of a_{ij},b_{ji} has degree one and the other has degree two. Hence we may write $f_n = \sum_{i=1}^q \alpha_i \beta_i$ for some linear forms α_i and quadratic forms β_i. Carrying out elementary transformations on the α_i, we may write $f_n = \sum_{i=1}^t \alpha'_i \beta'_i$ for some $t \leq q$, where $\alpha'_1, \ldots, \alpha'_t$ are linearly independent of the

form $\alpha'_i = Y_i - \gamma_i$ with γ_i linear forms in $Y_{t+1}, \ldots, Y_n$ $(1 \leq i \leq t)$.
Setting $Y_i = \gamma_i$, $1 \leq i \leq t$, we get the identity

$$\sum_{i=1}^{t} \gamma_i^3 + \sum_{i=t+1}^{n} Y_i^3 = 0.$$

But $n > 2t$, since $n > 2q \geq 2t$, so this is impossible. ■

Following 3.1, if A is a graded Cohen-Macaulay ring, we set

$$\mu(A) = \min\{\mu(M) \mid M \text{ a standard MCM } A\text{-module}\},$$

where $\mu(M)$ denotes the minimal number of generators of M.

Corollary 3.2 *For $n \geq 1$, $\mu(R_n) \geq n/2$.*

Proof: This follows immediately from 3.1 and the next remark 3.3(a). ■

Remark 3.3 *(a) Note that $\mu(R_2) = 1$ since there exist cyclic standard MCM modules over R_2, by the reducibility of f_2. Since f_3 is irreducible, $\mu(R_3) \geq 2$, so $\mu(R_3) = 2$ by 1.1.*
(b) W. Bruns has proved the following interesting result (see [He-Kü, (3.4)]): Suppose that M is a non-trivial MCM-module over a hypersurface ring R with isolated singularity. Then $2\cdot$ rank $M + 1 \geq$ dim R.
So, in our context, this gives the following fact:
If M is a standard MCM $R_n -$ module, then rank $M \geq (n-2)/2$.
In particular, $\mu(M) \geq (n-2)/2$, since $\mu(M) \geq$ rank$M$; this gives a weaker result than 3.2. On the other hand by a result of Herzog-Kühl (see [He-Kü, (1.3)]) we get $\mu(M) \geq 3\cdot$ rank $M/2$ for all standard MCM R_n-modules M and so $\mu(M) \geq 3(n-2)/4$, which is stronger than our 3.2 for $n \geq 8$.

The main result of this section (see 3.5 below) gives information on the growth of $\mu(R_n)$ with n. But first we have a preparatory lemma.

Lemma 3.4 *Let $n \geq 3$ and let q be a positive integer. Suppose that there exists no standard maximal Cohen-Macaulay R_n-module generated by q elements, but that there exists a standard maximal Cohen-Macaulay $R_{n-1}-$module minimally generated by q elements. Then there exist families of indecomposable standard maximal Cohen-Macaulay modules over R_n of rank q that are minimally generated by $2q$ elements.*

Proof: Let $\alpha \in K$ with $\alpha^3 \neq 1$ and consider the ring $R_\alpha = K[Y_1, \ldots, Y_{n-1}]/(f_{n-1} - \alpha^3 Y_{n-1}^3)$. Since clearly $R_\alpha \approx R_{n-1}$ as K-algebras, it follows from the hypotheses that there exists a standard MCM R_α-module M_α that is minimally generated by q elements. Let $(\phi_\alpha, \psi_\alpha)$ be the corresponding

matrix factorization over $K[Y_1,\ldots,Y_{n-1}]$.
Then

$$\widehat{\phi_\alpha} = \begin{pmatrix} \phi & (Y_n+Y_{n+1})I_q \\ -(Y_n^2 - Y_nY_{n+1} + Y_{n+1}^2)I_q & \psi \end{pmatrix},$$

,

$$\widehat{\psi_\alpha} = \begin{pmatrix} \psi & -(Y_n+Y_{n+1})I_q \\ (Y_n^2 - Y_nY_{n+1} + Y_{n+1}^2)I_q & \phi \end{pmatrix}$$

yields a reduced graded matrix factorization of a standard MCM module N_α over A_α, where $A_\alpha = K[Y_1,\ldots,Y_{n+1}]/(f_{n+1} - \alpha^3 Y_{n-1}^3)$. Thus $P_\alpha := N_\alpha/(Y_{n+1} - \alpha Y_{n-1})N_\alpha$ is a standard MCM module over $A_\alpha/(Y_{n+1} - \alpha Y_{n-1}) \approx R_n$ minimally generated by $2q$ elements. If P_α were decomposable (as standard MCM module), then P_α would have as direct summand a standard MCM module generated by q (or fewer) elements, contradicting our hypothesis.
It remains to show that rank $P_\alpha = q$.
First note that rank M_α+ rank $\Omega^1_{R_\alpha}(M_\alpha) = q$, because of the exact sequence

$$0 \to \Omega^1_{R_\alpha}(M_\alpha) \to R_\alpha^q \to M_\alpha \to 0.$$

Let $p \subseteq A_\alpha$ be a graded relevant prime ideal containing Y_n, Y_{n+1}, αY_{n-1} (recall that $n \geq 3$). Since $\alpha^3 \neq 1$, $f_{n+1} - \alpha^3 Y_{n-1}^3$ has an isolated singularity at the origin, so by the Auslander-Buchsbaum formula, $(N_\alpha)_p$ is free; hence $\mu((N_\alpha)_p) =$ rank $(N_\alpha)_p$.
Thus, setting $\bar{p} = p/(Y_n, Y_{n+1})$,

$$\mu((N_\alpha)_p) = \mu((N_\alpha/(Y_n,Y_{n+1})N_\alpha)_p) = \mu((M_\alpha)_{\bar{p}} \oplus (\Omega^1_{R_\alpha}(M_\alpha))_{\bar{p}})$$

$$= \mu((M_\alpha)_{\bar{p}}) + \mu((\Omega^1_{R_\alpha}(M_\alpha))_{\bar{p}}) = \text{rank } M_\alpha + \text{rank } \Omega^1_{R_\alpha}(M_\alpha) = q,$$

since $\{Y_n, Y_{n+1}\}$ is a regular sequence in A_α and $f_{n-1} - \alpha^3 Y_{n-1}^3$ has an isolated singularity at the origin, so that $(M_\alpha)_{\bar{p}}$ and $(\Omega^1_{R_\alpha}(M_\alpha))_{\bar{p}}$ are free over $(R_\alpha)_{\bar{p}}$.
Similarly,

$$\mu((N_\alpha)_p) = \mu((N_\alpha/(Y_{n+1} - \alpha Y_{n-1})N_\alpha)_p) = \mu((P_\alpha)_p) = \text{rank } P_\alpha,$$

since $(P_\alpha)_{pR_n}$ is free over $(R_n)_{pR_n}$. ■

We can now give the main results of this section:

Theorem 3.5 *Let $n \geq 3$ be such that $\mu(R_n) > \mu(R_{n-1})$. Then there exist families of standard indecomposable maximal Cohen-Macaulay modules over R_n of rank $\mu(R_{n-1})$ minimally generated by $2\mu(R_{n-1})$ elements. In particular, in this case $\mu(R_n) \leq 2\mu(R_{n-1})$.*

Proof: This follows immediately from 3.4. ■

Remark 3.6 *(a) The inequality $\mu(R_n) \le 2\mu(R_{n-1})$ is sharp, since for example $\mu(R_3) = 2$ while $\mu(R_2) = 1$.*
(b) Clearly $\mu(R_n) \ge \mu(R_{n-1})$ for all n. For if N is a standard MCM R_n-module with $\mu(N) = t$, N being presented by a graded matrix factorization (ϕ, ψ) of size t over R_n, then $\bar{N} := N/Y_nN$ is a standard MCM $R_{n-1}-$module with $\mu(\bar{N}) = t$, $\bar{N}$ being given by the matrix factorization $(\bar{\phi}, \bar{\psi})$, where the overbar denotes reduction modulo Y_n.

Remark 3.7 *Suppose in the statement of 3.4 that we drop the hypothesis about R_n having no standard MCM modules generated by q elements. Then, following the constructions in the proof of 3.4, by work of Yoshino [Yos1], if M_α is indecomposable over R_α then N_α is indecomposable over A_α. But it is not clear if the latter would force $P_\alpha := N_\alpha/(Y_{n+1} - \alpha Y_{n-1})N_\alpha$ to be indecomposable over R_n. This may be true for generic α. However if (ϕ, ψ) is a q-matrix factorization of an indecomposable MCM R_{n-1}-module then*

$$\begin{pmatrix} \phi & X_n^2 I_q \\ -X_n I_q & \psi \end{pmatrix}, \begin{pmatrix} \psi & -X_n^2 I_q \\ X_n I_q & \phi \end{pmatrix}$$

gives a $2q$-matrix factorization of an indecomposable standard MCM R_n-module by [Yos1].

Remark 3.8 *Note that $\mu(R_4) = 2$. As in Section 1, the decomposition*

$$f_4 = (Y_1 + Y_2)(Y_1^2 - Y_1Y_2 + Y_2^2) + (Y_3 + Y_4)(Y_3^2 - Y_3Y_4 + Y_4^2)$$

yields a 2-generated standard MCM R_4-module. As $\mu(R_4) < \mu(R_5)$ by 3.1, it follows from 3.5 that $3 \le \mu(R_5) \le 4$. However, if $\mu(R_5) = 3$, then, as in 2.4, there exists a standard MCM R_5-module of rank one. This contradicts 3.3(b). Hence $\mu(R_5) = 4$.

Remark 3.9 *There exists 3-generated standard MCM R_4-modules. Take*

$$\phi = \begin{pmatrix} Y_1 + Y_4 & Y_2 & Y_3 \\ Y_2 & -v^{-1}Y_3 & -u(Y_1 + \rho^2 Y_4) \\ vY_3 & u^{-1}(Y_1 + \rho Y_4) & -Y_2 \end{pmatrix}$$

where $\rho, u, v \in K \setminus \{0\}$ satisfy

$$\rho^3 = 1, \rho \ne 1;$$

$$-uv + u^{-1} + v^{-1} = 0;$$

$$-uv + u^{-1}\rho^2 + v^{-1}\rho = 0;$$

that is, $u^3 = \rho(1 + \rho)$ and $v = -u/(1 + \rho)$.
Then $\det \phi = f_4$, so $(\phi, adj(\phi))$ give a reduced graded matrix factorization over $K[Y_1, Y_2, Y_3, Y_4]$.

References

[Br-He] W. Bruns and J. Herzog, *Cohen-Macaulay rings.* Cambridge University Press, Cambridge, 1993.

[Eis] D. Eisenbud, *Homological algebra with an application to group representations.* Trans. American Math. Soc. **260** (1980), 35-64.

[He-Kü] J. Herzog and M. Kühl, *Maximal Cohen-Macaulay modules over Gorenstein rings and Bourbaki sequences.* In *Commutative algebra and combinatorics.* (Kyoto, 1985), 65-92, Adv. Stud. Pure Math. **11**, North-Holland, Amsterdam-New York, 1987.

[Ka] C.P. Kahn, *Reflexive modules on minimally elliptic singularities.* Math. Ann. **285** (1989), 141-160.

[OP] L. O'Carroll and D. Popescu, *Free resolutions for deformations of maximal Cohen-Macaulay modules.* Comm. in Algebra, to appear.

[Yos1] Y. Yoshino, *Tensor product of matrix factorizations.* Nagoya Math. J. **152** (1998), 39-56.

[Yos2] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings.* London Math. Soc. Lecture Notes Ser. **146**, Cambridge, 1990.

[1] *Department of Mathematics, University of Kaiserslautern*
Kaiserslautern 67653, Germany
E-mail: laza@mathemtik.uni-kl.de
[2] *Department of Mathematics and Statistics, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, Scotland*
E-mail: loc@maths.ed.ac.uk
[3] *Institute of Mathematics, University of Bucharest,*
P.O.Box 1-764, Bucharest 70700, Romania
E-mail: dorin@stoilow.imar.ro