STRUCTURE AND REPRESENTATIONS OF CONFORMAL ALGEBRAS

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Introduction

0.1. In the last two decades vertex algebras have been an important tool in such diverse subjects as representations of infinite-dimensional algebras and the theory of finite groups [15, 19].

Roughly speaking, a vertex algebra is a space V such that to each element of V there corresponds a *formal distribution*, i.e. an element of $\operatorname{End} V[[z^{\pm 1}]]$. (Note that any algebraic operation performed on the space of formal distribution will have to involve more than one variable, as the distributions are power series in both z and z^{-1}.) Two distributions $a(z)$ and $b(w)$ must be *local*, that is, commute outside the diagonal of the zw-plane.

0.2. The first definition of vertex algebras was given in [6] and is rather involved. With time a need for an algebraic formalism for vertex algebras became clear. Since local formal distributions are in some sense meromorphic, it is reasonable to look at the "singular" part first. Such an approach was emphasized in [26, 27] and especially [19], where this theory was fully developed (see also [4, 23] for geometric counterparts).

This setting is quite general; for an algebra A, consider the following operation on formal distributions over A:

$$a(z) \circledn b(z) = \operatorname{Res}_{w=0} a(w)b(z)(w-z)^n, \quad a(z), b(z) \in A[[z^{\pm 1}]], n \in \mathbb{Z}_{\geq 0}. \tag{0.1}$$

Two formal distributions are local if a finite number of their products (0.1) is nonzero. The formalization of this definition leads to the concept of a *conformal algebra* (Definition 1.1). Then a vertex algebra is defined as having two related structures: that of a Lie conformal algebra and a left symmetric differentiable algebra [2].

0.3. Conformal algebras also have an intriguing connection to Hamiltonian formalism in the theory of nonlinear evolutionary equations [1, 34]. In fact the first appearance in the literature of conformal-like structures predates the discovery of vertex algebras and comes from the calculus of variations [17]. However, this subject is outside the scope of this survey.

0.4. This survey is dedicated to the study of conformal algebras and their representations. Other expository papers on the subject appeared in the past, in particularly, [20] and [35]; however, there have been new developments in the field since their publication.

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We begin by defining conformal algebras and discussing their general properties in Chapter 1. It also contains several basic examples. Chapter 2 develops the theory of representations of conformal algebras. The bulk of this survey, Chapter 3, is dedicated to the very important conformal algebras gc_n and Cend_n and their subalgebras (roughly, they are the analogs of matrix algebras). We conclude with conjectures and open questions in Chapter 4.

Most of the results here appeared elsewhere; however, several remarks are new.

Throughout this survey the base field is $\mathbb{C}$.

1. Basic Definitions and Examples

1.1. We begin with the formal definition of a conformal algebra.

Definition 1.1. [19] A *conformal algebra* C is a $\mathbb{C}[\partial]$-module endowed with bilinear operations $\textcircled{n} : C \otimes C \to C$, $n \in \mathbb{Z}_{\geq 0}$, such that for any $a, b \in C$

(1) (locality axiom) $a \,\textcircled{n}\, b = 0$ for $n > N(a, b)$
 ($N(a, b)$ is called the *order of locality* of a and b);
(2) (Leibniz rule) $\partial(a \,\textcircled{n}\, b) = (\partial a) \,\textcircled{n}\, b + a \,\textcircled{n}\, (\partial b)$;
(3) $(\partial a) \,\textcircled{n}\, b = -na \;\; {}_{n-1} \;\; b$.

A more succinct way to present operations in a conformal algebra C is via the so-called λ-*product.* Let λ be a formal variable. Define the map $C \otimes C \to \mathbb{C}[\lambda] \otimes C$ as

$$a_\lambda b = \sum_n \frac{\lambda^n}{n!} a \,\textcircled{n}\, b, \quad a, b \in C, n \in \mathbb{Z}_{\geq 0}. \tag{1.1}$$

We then arrive at an alternative definition of a conformal algebra: this is a $\mathbb{C}[\partial]$-module with a bilinear λ-product satisfying analogs of axioms (2) and (3) (a quick exercise is to deduce them explicitly!). Clearly locality is automatic here; after all, the λ-product produces a polynomial in λ. The λ-product works extremely well in calculations; we will see some evidence of this below.

1.2. A set of mutually local formal distributions closed with respect to product $\textcircled{n}$ and ∂_z is a conformal algebra. Thus conformal algebras provide an algebraic formalism for algebras of local formal distributions. Conversely, every conformal algebra can be made into an algebra of formal distributions. This is the procedure:

Let C be a conformal algebra. For each integer n, consider the linear space $\hat{A}(n)$ isomorphic to C. The element corresponding to $a \in C$ is denoted $\hat{a}(n)$. Let $\hat{A} = \bigoplus_{n \in \mathbb{Z}} \hat{A}(n)$ and let E be the subspace of $\hat{A}$ spanned by the elements of the form $(\partial a)(n) + na(n-1)$ for all $a \in C, n \in \mathbb{Z}$. The quotient space $\operatorname{Coeff} C = \hat{A}/E$ is the *coefficient algebra* of C. The image $\hat{a}(n)$ in $\operatorname{Coeff} C$ is denoted $a(n)$.

It remains to introduce the operation on $\operatorname{Coeff} C$. The following formula holds for any two formal distributions a and b:

$$a(m)b(n) = \sum_{j \geq 0} \binom{m}{j} (a \;\; {}_j \;\; b)(m + n - j).$$

Now we take it as the definition of the product in $\operatorname{Coeff} C$. It follows that C is isomorphic to an algebra of formal distributions over $\operatorname{Coeff} C$ (and ∂ acts properly because we factored out E).

Remark 1.2. The above construction is taken from [31]. [19] follows a somewhat different approach by introducing a conformal structure on $\hat{A}$. (This is more similar to the original construction for vertex algebras in [6].)

Coeff C is universal among all algebras of possible coefficients. Namely, let B be an algebra such that there exists a homomorphism $C \to B[[z, z^{-1}]]$. Then there exists a unique homomorphism $\phi : \mathrm{Coeff}\, C \to B$ such that the diagram

$$\begin{array}{ccc} \mathrm{Coeff}\, C[[z, z^{-1}]] & \xrightarrow{\phi} & B[[z, z^{-1}]] \\ & \nwarrow \quad \nearrow & \\ & C & \end{array}$$

commutes.

1.3. Now we can define Lie and associative conformal algebras.

Let $\mathcal{X}$ be a variety of algebras (Lie, associative, commutative, etc). Then we say that C is $\mathcal{X}$ *conformal* if Coeff C is $\mathcal{X}$ (or more rigorously lies in $\mathcal{X}$).

This definition is in a sense unsatisfactory: it refers to another object and a non-conformal one at that. An improvement would be to define a variety of conformal algebras directly. So, assume that $\mathcal{X}$ is defined by identities $\{f_\alpha\}$ (e.g. the Jacobi identity and anti-commutativity for Lie algebras). Then one can produce identities $\{g_\alpha\}$ such that a conformal algebra C satisfies them if and only if C is $\mathcal{X}$ conformal. The algorithm for the construction of such identities g_α can be found in [19]; a more detailed discussion appears in [24].

Conformal identities look much better in their λ-form. To provide a few examples, the conformal law of associativity is (for any $a, b, c \in C$)

$$a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c, \tag{1.2}$$

and anti-commutativity and the Jacobi identity are, respectively,

$$[a_\lambda b] = -[b_{-\partial-\lambda} a], \tag{1.3}$$

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]] \tag{1.4}$$

(we denote the λ-product in a Lie conformal algebra as $[\ {}_\lambda\ ]$ to emphasize its relation to the ordinary Lie bracket. That is, for a Lie conformal algebra C we have $[\ {}_\lambda\ ] : C \otimes C \to C[\lambda]$).

Remark 1.3. An associative conformal algebra can always be turned into a Lie conformal one. The λ-bracket is defined as

$$[a_\lambda b] = a_\lambda b - b_{-\lambda-\partial} a.$$

This allows us to abuse the language sometimes and speak of Lie subalgebras of an associative conformal algebra C without mentioning that we first turn C into a Lie conformal algebra as above. The same goes for embeddings of Lie conformal algebras into associative ones etc.

At this point a reasonable question would be, If we have something like "conformal varieties", can we also get free conformal algebras? In such generality, the answer is negative. Indeed, a free object must map onto any other, and for conformal algebras this means that the orders of locality for generators in a free conformal algebra would be unbounded. However, if we restrict the orders of locality for generators beforehand, we can construct a free conformal algebra. Namely, let S be a set of letters with a given function $N : S \times S \to \mathbb{Z}_{\geq 0}$. Consider the category of

$\mathcal{X}$ conformal algebras generated by S such that for any $a, b \in S$ and $n > N(a, b)$, $a \textcircled{n} b$=0. This category does possess the universal object called the *free conformal algebra* corresponding to the function N. For details, see [31] or [5].

The above discussion shows that a conformal algebra can be described as universal in some category only if all objects in this category satisfy given locality conditions. With this in mind, one can ask if there exist universal enveloping algebras for Lie conformal algebras. This turns out to be true for some algebras (e.g. finite) but in general there exist Lie conformal algebras that can not be embedded into associative conformal algebras (in the sense of Remark 1.3), see [32].

1.4. The paragraphs above dealt with "universal algebra"; we continue with the definitions from the structure theory.

A *subalgebra* of a conformal algebra C is a $\mathbb{C}[\partial]$-submodule of C closed with respect to all the operations $\textcircled{n}$. An *ideal* of C is a $\mathbb{C}[\partial]$-submodule I such that for all n, $C \textcircled{n} I \subset C$ and $I \textcircled{n} C \subset C$.

A conformal algebra C is *simple* if its only ideals are C and 0.

A conformal algebra C is *nilpotent* of order d if for any $n_1, n_2, \dots, n_{d-1} \in \mathbb{Z}_{\geq 0}$, $C \ {}_{n_1}\ C \dots \ {}_{n_{d-1}}\ C = 0$. (There is an abuse of notation going on: the product of d elements of C is not defined unless we say how the brackets are inserted. Here we mean that the product is 0 for any insertion of brackets.)

An associative conformal algebra C is *semisimple* if it has no nonzero nilpotent ideals.

The *derived conformal algebra* of a Lie conformal algebra C is $C' = \sum_n C \textcircled{n} C$. As usual we set $C^{(1)} = C'$, $C^{(m+1)} = (C^{(m)})'$, $m \geq 1$, and say that C is *solvable* if $C^{(m)} = 0$ for some m. A Lie conformal algebra is *semisimple* if it has no nonzero solvable ideals.

One can go on and define, e.g., prime conformal algebras, various conformal radicals, and so on, but we do not need them in this survey.

1.5. Now we turn to examples of conformal algebras.

The first example shows that every "ordinary" algebra can be made conformal.

Example 1.4. Let B be an algebra. Consider the affinization $B[t^{\pm 1}]$ of B and the collection $\mathcal{F} \subset B[t^{\pm 1}][[z, z^{-1}]]$ of formal distributions of the form

$$\widetilde{b} = \sum_{m \in \mathbb{Z}} b t^m z^{-m-1}, \quad b \in B. \tag{1.5}$$

We claim that the module $\operatorname{Cur} B = \mathbb{C}[\partial] \otimes \mathcal{F}$ is a conformal algebra. For this we have only to show that all elements of $\operatorname{Cur} B$ are pairwise mutually local.

It is easy to see that for any $a, b \in B$,

$$\widetilde{a} \textcircled{0} \widetilde{b} = \widetilde{ab}, \qquad \widetilde{a} \textcircled{n} \widetilde{b} = 0, \ n > 0$$

(where the products $\textcircled{n}$ are understood in the sense of (0.1)), and as for the rest of the elements of $\operatorname{Cur} B$, their mutual locality follows from axioms (2) and (3).

$\operatorname{Cur} B$ is called the *current algebra* over B.

Remark 1.5. It was easy to check mutual locality for all elements of $\operatorname{Cur} B$; however, this might not be so for an arbitrary collection of formal distributions. Fortunately, there is a way out for Lie and associative algebras.

Lemma 1.6 (Dong's Lemma). *Let $a, b,$ and c be pairwise mutually local formal distributions over a Lie or associative algebra. Then for all $n \in \mathbb{Z}_{\geq 0}$ the formal distributions $a \circled{n} b$ and c are mutually local.*

In particular, Dong's lemma implies that for Lie and associative conformal algebras we have only to check that the generators are local.

For associative algebras, there exists a generalization of current algebras:

Example 1.7. Let B be an associative algebra with a locally nilpotent derivation δ. Consider its (localized) Ore extension $B[t^{\pm 1}; \delta]$ and the collection $\mathcal{F} \subset B[t^{\pm 1}; \delta][[z, z^{-1}]]$ of formal distributions of the type (1.5). For $a, b \in B$, we have

$$\widetilde{a} \circled{n} \widetilde{b} = (-1)^n \widetilde{a\delta^n(b)}$$

(or $\widetilde{a}_\lambda \widetilde{b} = \widetilde{ae^{-\lambda\delta}b}$ in the λ-notation).

The conformal algebra $\operatorname{Diff} B = \mathbb{C}[\partial] \otimes \mathcal{F}$ is called the *differential algebra* over B.

In the Lie case, the smallest non-current conformal algebra is the Virasoro conformal algebra:

Example 1.8. Consider the Lie algebra $\operatorname{Vect} \mathbb{C}^\times$ of regular vector fields on $\mathbb{C}^\times$. It is well known that the fields $L_n = -t^{n+1}\partial_t$, $n \in \mathbb{Z}$, form a basis of $\operatorname{Vect} \mathbb{C}^\times$. The formal distribution $L(z) = \sum_n L_n z^{-n-2}$ is local with itself:

$$[L_\lambda L] = (\partial + 2\lambda)L.$$

The conformal algebra $\operatorname{Vir} = \mathbb{C}[\partial] \otimes L$ is called the *Virasoro conformal algebra.*

Remark 1.9. $\operatorname{Vect} \mathbb{C}^\times$ is the algebra of infinitesimal conformal transformations of $\mathbb{C}^\times$. This explains the choice of the term "conformal" since Vir is the smallest non-trivial (i.e. non-current) conformal algebra.

Of course, the "ordinary" Virasoro algebra is not $\operatorname{Vect} \mathbb{C}^\times$ but its central extension. However, one can also construct the central extension $\widehat{\operatorname{Vir}}$ of Vir such that $\operatorname{Coeff} \widehat{\operatorname{Vir}}$ is the Virasoro algebra:

Example 1.10. As a vector space $\widehat{\operatorname{Vir}} = \operatorname{Vir} \oplus \mathbb{C}\mathsf{c}$ (here $\partial \mathsf{c} = 0$) and the λ-brackets are

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}\mathsf{c}, \quad [L_\lambda \mathsf{c}] = 0, \quad [\mathsf{c}_\lambda \mathsf{c}] = 0.$$

In the same vein, for a simple finite-dimensional Lie algebra $\mathfrak{g}$, one can construct the central extension $\widehat{\operatorname{Cur} \mathfrak{g}}$ of $\operatorname{Cur} \mathfrak{g}$ such that $\operatorname{Coeff} \widehat{\operatorname{Cur} \mathfrak{g}} = \widehat{\mathfrak{g}}$. Namely, $\widehat{\operatorname{Cur} \mathfrak{g}}$ is generated by elements $\widetilde{g}$, $g \in \mathfrak{g}$, and c (again, $\partial \mathsf{c} = 0$) such that

$$[\widetilde{g}_\lambda \widetilde{h}] = \widetilde{[g, h]} + \lambda(g|h)\mathsf{c}, \quad [\widetilde{g}_\lambda \mathsf{c}] = 0, \quad [\mathsf{c}_\lambda \mathsf{c}] = 0.$$

1.6. For brevity we say that a conformal algebra is *finite* if it is finite as a $\mathbb{C}[\partial]$-module.

A current algebra $\operatorname{Cur} B$ is simple (respectively, finite) if and only if B is simple (respectively, finite). Thus, we already know a number of (admittedly not very interesting) examples of finite simple conformal algebras. Are there others?

In the Lie case, the Virasoro conformal algebra is also simple and finite but this is it:

Theorem 1.11. (1) [11] *Let C be a finite simple Lie conformal algebra. Then C is isomorphic to either* Vir *or a current algebra* Cur $\mathfrak{g}$ *over a simple finite-dimensional Lie algebra* $\mathfrak{g}$.

(2) [20] *Let C be a finite simple associative conformal algebra. Then C is isomorphic to a current algebra* $\operatorname{Cur}\operatorname{Mat}_n(\mathbb{C})$.

The first part of Theorem 1.11 is proved by the careful study of the Lie algebra of non-negative coefficients of C. When completed with respect to its natural topology, this algebra becomes linearly compact and then, after some additional work, one applies the Cartan–Guillemin theorem to obtain the complete classification.

The second part follows from the first via standard algebraic techniques.

Remark 1.12. As we have just seen, in the conformal setting the (centerless) Virasoro algebra and the Laurent extensions of finite-dimensional simple Lie algebras appear as coefficient algebras of finite simple Lie conformal algebras. In the non-conformal universe, however, these belong to two very distinct worlds of Cartan and affine Kac–Moody algebras.

We can extend Theorem 1.11 to the semisimple case but it is not as straightforward as one may think:

Example 1.13. For a finite-dimensional Lie algebra $\mathfrak{g}$, the $\mathbb{C}[\partial]$-module $\operatorname{Vir}\oplus\operatorname{Cur}\mathfrak{g}$ carries the following conformal structure:

$$[L_\lambda L] = (\partial + 2\lambda)L, \quad [\widetilde{g}_\lambda \widetilde{h}] = \widetilde{[g,h]}, \quad [L_\lambda \widetilde{g}] = (\partial + \lambda)\widetilde{g}.$$

So, we obtain the semidirect product of Vir and Cur $\mathfrak{g}$. When $\mathfrak{g}$ is semisimple, this semidirect product is semisimple too.

This, however, is the only surprise in the classification of finite semisimple Lie conformal algebras:

Theorem 1.14. (1)[11] *Let C be a finite semisimple conformal Lie algebra. Then C is isomorphic to a direct sum of copies of finite simple Lie conformal algebras and semidirect products of* Vir *and* Cur $\mathfrak{g}$ *for semisimple finite-dimensional* $\mathfrak{g}$*'s.*

(2)[20] *A finite semisimple associative conformal algebra is isomorphic to a direct sum of copies of simple associative conformal algebras.*

The second part of this theorem seems to be a weak version of the Artin–Wedderburn theorem: in the end, all we get is matrices. However, from the representation-theoretic point of view, current algebras over matrices and, in general, finite conformal algebras are not the right analogs of ordinary matrices. We discuss this in the next chapter.

2. Representation Theory of Conformal Algebras

Defining a module M over a conformal algebra C is easy and we now have a choice of two approaches: either modify Definition 1.1 by considering the products $C \otimes M \to M$ or define a module of formal distributions over an algebra of formal distributions imitating (0.1). However, it is even better to take a more general approach and start with the definition of a conformal linear map. In particular, this will help us to construct certain representations later on.

2.1. Let M and N be two $\mathbb{C}[\partial]$-modules. A *conformal linear map* from M to N is a $\mathbb{C}$-linear map $\phi : M \to \mathbb{C}[\lambda] \otimes N$, denoted $\phi_\lambda : M \to N$, such that $\partial\phi_\lambda - \phi_\lambda\partial = -\lambda\phi_\lambda$. As above, we can define the operators $\phi_{(n)}$ via $\phi_\lambda m = \sum_n (\lambda^n/n!)\phi_{(n)} m$.

We can "differentiate" conformal linear maps by putting $(\partial\phi)_\lambda = -\lambda\phi_\lambda$.

The space of all conformal linear maps from M to N is denoted $\mathrm{Chom}(M, N)$. It carries a natural structure of a $\mathbb{C}[\partial]$-module. When $M = N$, we can also define the products $\psi_{(n)}\phi$ of conformal linear maps by applying the conformal law of associativity (1.2). However, in general this does not make $\mathrm{Chom}(M, M)$ into a conformal algebra as the locality condition fails. For instance, let M be of infinite rank and $\phi \in \mathrm{Chom}(M, M)$ be such that for any n there exists $u_n \in M$ whose order of locality with ϕ is n. Put $v_n = \phi_{(n)} u_n$. Let $\psi \in \mathrm{Chom}(M, M)$ be such that $\psi_{(0)} v_n \neq 0$ for all n. Then for any n, $(\psi_{(n)}\phi)_{(n)} u_n \neq 0$ and thus ψ and ϕ are not local.

On the other hand, if M is of finite rank, any two elements of $\mathrm{Chom}(M, M)$ are local with each other. In this case we denote the associative conformal algebra $\mathrm{Chom}(M, M)$ as $\mathrm{Cend}\, M$.

$\mathrm{Cend}\, M$ with a Lie conformal bracket (see Remark 1.3) is denoted $\mathrm{gc}\, M$.

To simplify notations, we also denote $\mathrm{Cend}\, \mathbb{C}[\partial]^n$ and $\mathrm{gc}\, \mathbb{C}[\partial]^n$ as Cend_n and gc_n, respectively.

2.2. Now we will define a module over a conformal algebra.

Definition 2.1. A *module* M over a Lie or associative conformal algebra C is a $\mathbb{C}[\partial]$-module endowed with an operation $C \otimes M \to \mathbb{C}[\lambda] \otimes M$ such that for any $a, b \in C$ and $v \in M$,

(1) $(\partial a)_\lambda v = -\lambda a_\lambda v, \quad a_\lambda \partial v = (\partial + \lambda)(a_\lambda v)$;
(2) $a_\lambda(b_\mu v) = [a_\lambda b]_{\lambda+\mu} v + b_\mu(a_\lambda v)$ if C is Lie;
$a_\lambda(b_\mu v) = (a_\lambda b)_{\lambda+\mu} v$ if C is associative.

Simply put, M is a C-module if there exists a map $C \to \mathrm{Chom}(M, M)$ of conformal algebras that satisfies a version of the Jacobi identity or associativity.

As usual we can define a *submodule*, an *irreducible* module (contains no non-trivial submodules), an *indecomposable* module (does not split into a direct sum of non-trivial submodules), etc. We call a module *finite* if it is finite over $\mathbb{C}[\partial]$.

Remark 2.2. If C is a Lie conformal algebra and M and N are modules over C, $\mathrm{Chom}(M, N)$ also carries a natural structure of a C-module. Namely, we set

$$(a_\lambda\phi)_\mu m = a_\lambda(\phi_{\mu-\lambda} m) - \phi_{\mu-\lambda}(a_\lambda m), \quad a \in C, \phi \in \mathrm{Chom}(M, N), m \in M.$$

Then we can define the *contragradient* C-module $U^* = \mathrm{Chom}(M, \mathbb{C})$, where $\mathbb{C}$ stands for the trivial C-module (with the trivial action of ∂).

For a finite M, we also can define $M \otimes N = \mathrm{Chom}(M^*, N)$.

As in Chapter 1, one can show that a module over a conformal algebra C can be always viewed as a module of formal distributions over a "coefficient module" which, in turn, is a module over $\mathrm{Coeff}\, C$.

2.3. Below we construct modules for finite simple conformal algebras (cf. Theorem 1.11).

Example 2.3. For any $\Delta, \alpha \in \mathbb{C}$ consider the space of densities $F(\Delta, \alpha) = \mathbb{C}[t, t^{-1}]e^{-\alpha t}(dt)^{1-\Delta}$. This is naturally a module over $\mathrm{Vect}\, \mathbb{C}^\times$; it is irreducible whenever $\Delta \neq 0$.

The formal distribution $m(z) = \sum_n (t^n e^{-\alpha t}(dt)^{1-\Delta})z^{-n-1}$ spans the module $M(\Delta, \alpha)$ over the Virasoro conformal algebra Vir with the action induced from that of $\operatorname{Vect}\mathbb{C}^\times$ on $F(\Delta, \alpha)$. Explicitly, $L_\lambda m = (\partial_z + \alpha + \Delta\lambda)m$. Again, whenever $\Delta \neq 0$, $M(\Delta, \alpha)$ is irreducible.

In fact, the following is true:

Theorem 2.4. [9] *Any non-trivial irreducible finite module over the Virasoro conformal algebra is isomorphic to $M(\Delta, \alpha)$ with $\Delta \neq 0$.*

Finite indecomposable modules over Vir were studied in [10]. Complete reducibility does not hold here. In fact, by classifying finite central extensions of $M(\Delta, \alpha)$, one can arrive directly at the classification of central extensions of certain modules over regular vector fields on $\mathbb{C}$. This is an example of a connection between the cohomology of conformal algebras [3] and the cohomology of infinite-dimensional Lie algebras [14, 13, 18].

Example 2.5. It is easy to construct a module over a current algebra. Let A be an algebra and U an A-module. Then $\operatorname{Cur} A$ acts on the module $\widetilde{U} = \mathbb{C}[\partial] \otimes U$ with the natural action $\widetilde{a}_\lambda(1 \otimes u) = 1 \otimes au$, $a \in A$, $u \in U$.

A companion result to Theorem 2.4, also proved in [9], states that a non-trivial irreducible finite module over $\operatorname{Cur}\mathfrak{g}$ for a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ is of the form $\widetilde{U}$ for a non-trivial irreducible finite-dimensional $\mathfrak{g}$-module U. We will see below (Theorem 3.10) that an even more general result holds for associative unital algebras: every module over such an algebra is of the form $\widetilde{U}$ (and $\widetilde{U}$ is irreducible if and only if U is).

3. Cend_n and gc_n

3.1. Here we present the explicit constructions of the conformal algebras Cend_n and gc_n.

Every conformal endomorphism of the module $\mathbb{C}[\partial]$ is determined by the image of ∂, thus roughly speaking, Cend_1 is isomorphic to $\mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathbb{C}[\partial]$ (where the second component is responsible for the image and the first, for the $\mathbb{C}[\partial]$-module structure of this conformal algebra). In the case of $\mathbb{C}[\partial]^n = \mathbb{C}[\partial] \otimes \mathbb{C}^n$ we have also to account for the action of $\mathrm{Mat}_n(\mathbb{C})$ on $\mathbb{C}^n$.

However, to get more explicit expressions, it is perhaps better to go down to the level of coefficients.

The coefficient algebra of Cend_n or gc_n is the algebra $\mathrm{Mat}_n(\mathcal{D}(\mathbb{C}^\times))$ viewed as either an associative or Lie algebra. (Here $\mathcal{D}(M)$ denotes the algebra of differential operators on M.) The formal distributions that span Cend_n as a $\mathbb{C}[\partial]$-module (here and further, we simply write ∂ for ∂_z) are

$$J_A^m = \sum_{n \in \mathbb{Z}} At^n(-\partial_t)^m z^{-n-1}, \quad A \in \mathrm{Mat}_n(\mathbb{C}), m \in \mathbb{Z}_{\geq 0}.$$

The action on $\mathbb{C}[\partial]^n$ arises from the standard action of $\mathrm{Mat}_n(\mathcal{D}(\mathbb{C}^\times))$ on the space $\mathbb{C}^n[t^{\pm 1}]$. Namely, for $v \in \mathbb{C}^n$ let $\tilde{v} = \sum_n vt^n z^{-n-1}$. Then

$$J_A^m{}_\lambda \tilde{v} = (\partial + \lambda)^m Av. \tag{3.1}$$

We call this action *canonical*.

Remark 3.1. We can tweak the above construction by putting

$$J_A^m{}_\lambda \tilde{v} = (\partial + \lambda + \alpha)^m A v, \quad \alpha \in \mathbb{C}.$$

More explicitly, we consider the action of $\mathrm{Mat}_n(\mathcal{D}(\mathbb{C}^\times))$ on the space $\mathbb{C}^n[t^{\pm 1}]e^{-\alpha t}$ and then pass to the formal distributions. Notice, though, that we still get an action of Cend_n on $\mathbb{C}[\partial]^n$. We denote this representation E_n^α.

In fact, this action comes from the automorphism of Cend_n sending the generator J_A^1 to $J_A^1 + \alpha J_{\mathrm{Id}}^0$, where Id is the identity matrix.

Another way to obtain the explicit presentation of Cend_n is to use the language of differential conformal algebras (see Example 1.7). (This is more in tune with our statement that $\mathrm{Cend}_1 = \mathbb{C}[\partial] \otimes \mathbb{C}[\partial]$.) Here we have $\mathrm{Cend}_n = \mathrm{Diff}(\mathbb{C}[\partial_t] \otimes \mathrm{Mat}_n(\mathbb{C}))$ for the derivation $\mathrm{ad}\, t = [t, \cdot\,]$.

Actually, an easy computation shows that both explicit presentations above produce the same algebra of formal distributions; we just get two different bases, $\{J_A^m\}$ and $\{\widetilde{f(\partial_t)}\}$. Though the formula for the action of the second basis on $\mathbb{C}[\partial]^n$ does not look as good as (3.1), it still has merits (see below).

Remark 3.2. The coefficient algebra of gc_n can be embedded into the algebra $\widehat{\mathrm{gl}_\infty}$ (a central extension of the algebra of infinite-dimensional matrices with finitely many nonzero diagonals). This relation is used in the study of finite modules of gc_n but its precise nature–as well as its connection to vertex algebras–is outside the scope of this survey.

3.2. It is clear from the construction that Cend_n contains $\mathrm{Cur}\,\mathrm{Mat}_n(\mathbb{C})$, i.e. that every simple finite conformal algebra is contained in Cend_n for some n. And, for every simple finite-dimensional Lie algebra $\mathfrak{g}$, $\mathrm{Cur}\,\mathfrak{g}$ embeds into gc_n for some n.

The construction of Cend_n (and thus gc_n) also implies that gc_n contains the Virasoro conformal algebra Vir. Moreover, for every $\alpha \in \mathbb{C}$, the conformal subalgebra of gc_n generated by $L_\alpha = J_{\mathrm{Id}}^1 + \alpha J_{\mathrm{Id}}^0$ is isomorphic to Vir (cf. Remark 3.1).

It is more useful here to pass to the basis $\{\widetilde{f(\partial_t)}\}$ of gc_n. In the above notation, $L_\alpha = \widetilde{\partial}_t + \partial\widetilde{1} + \alpha\widetilde{1}$. We can go further and consider the element $L_{\alpha,\beta} = \widetilde{\partial}_t + \beta\partial\widetilde{1} + \alpha\widetilde{1}$ for any $\beta \in \mathbb{C}$. The subalgebra generated by $L_{\alpha,\beta}$ is also isomorphic to Vir.

This gives us a family of embeddings $\theta_{\alpha,\beta} : \mathrm{Vir} \hookrightarrow \mathrm{gc}_1$. It is not difficult to show that every embedding of Vir into gc_1 is of this form. Moreover, a map $\theta_{\alpha,\beta}$ and the canonical gc_1-action on $\mathbb{C}[\partial]$ establish the isomorphism $\mathbb{C}[\partial] \simeq M(1-\beta, \alpha)$ (see Example 2.3). Thus we have just classified all Vir modules of rank one.

Remark 3.3. A direct classification of all embeddings of Vir into gc_n for an arbitrary n would be of great interest. Among other things it would imply the classification of Virasoro elements in gc_n and the complete description of finite Vir modules.

3.3. We essentially view conformal algebras Cend_n and gc_n as analogs of matrix algebras. Since the theory of finite associative algebras is much simpler than its Lie counterpart, we should first focus on the study of Cend_n. There are three directions here: the representation theory, a purely algebraic description of Cend_n (i.e. the first cornerstone for the analog of Artin–Wedderburn), and a more detailed study of its subalgebras. The latter approach, of course, is a dead end when ordinary matrix algebras are concerned but we already saw that Cend_n possesses some interesting subalgebras.

3.4. We begin by focusing on a possible analog of the Artin–Wedderburn theorem.

The biggest problem is that unlike $\mathrm{Mat}_n(\mathbb{C})$, Cend_n is not of finite rank. This suggests to develop a concept of growth for conformal algebras. In analogy with the ordinary theory [25], we define the *Gelfand–Kirillov dimension* of a finitely-generated conformal algebra C as

(3.2)
$$\mathrm{GKdim}\, C = \limsup_{r\to\infty} \frac{\log \mathrm{rk}_{\mathbb{C}[\partial]}(\mathbb{C}[\partial]\text{-span of products of} \le r \text{ generators of } C)}{\log r}.$$

By locality all the ranks in (3.2) are finite and, since the function we consider here is monotone, $\mathrm{GKdim}\, C$ is well defined. It possesses the standard properties: GKdim is does not depend on the choice of the generating set, GKdim of a subalgebra or a quotient algebra does not exceed that of the algebra, etc.

For any differential conformal algebra $\mathrm{Diff}\, B$, $\mathrm{GKdim}\,\mathrm{Diff}\, B = \mathrm{GKdim}\, B$. In particular, $\mathrm{GKdim}\,\mathrm{Cend}_n = 1$.

Remark 3.4. In general, for an associative conformal algebra C, $\mathrm{GKdim}\,\mathrm{Coeff}\, C \le \mathrm{GKdim}\, C + 1$ [28] and the inequality is sometimes strict (e.g. when C is torsion). It is still an open question if the equality is always reached for a torsion-free conformal algebra.

So far we can say that Cend_n is simple, of GKdimension 1, and differential. The latter property, however, refers to the coefficient algebra and we wish to obtain a description of Cend_n in purely conformal terms.

Mimicking the ordinary algebra, we have to start by defining unital conformal algebras. An ordinary unital algebra contains the field (i.e. a subalgebra of dimension one) whose action is nonzero. The only non-trivial conformal algebra of rank one is $\mathrm{Cur}\,\mathbb{C}$ and, moreover, every $\mathrm{Cur}\,\mathbb{C}$ module M splits as $M = M^0 \oplus M^1$, where $\widetilde{1}\,⓪ = \mathrm{Id}_{M^1}$ and the action on M^0 is zero (both facts are not hard to show). Hence, we will call an associative conformal algebra C unital if it contains $\mathrm{Cur}\,\mathbb{C}$ and if for the resulting action of $\mathrm{Cur}\,\mathbb{C}$, $C = C^1$.

The element $\widetilde{1}$ is called a *conformal identity.*

Remark 3.5. A more rigorous name would be a *left* conformal identity as we have only the left action of $\mathrm{Cur}\,\mathbb{C}$ on C. And, unlike, an ordinary (two-sided) identity, a conformal identity is not unique.

A differential algebra $\mathrm{Diff}\, B$ can be always made unital by adjoining identity to B. (It is still unknown if one can adjoin a conformal identity to a torsion-free conformal algebra.) The converse is almost true.

Let the *left annihilator* of C be the set $\{a \ni C \mid a_\lambda C = 0\}$.

Theorem 3.6. [28] *An associative conformal algebra C with the zero left annihilator is differential, $C = \mathrm{Diff}\, B$. Also, if C is finitely generated, then so is B.*

(The main part of the proof is to show that the conformal structure of C is encoded by the zeroth coefficients and $\widetilde{1}(1)$. It immediately follows that the coefficient algebra is an Ore extension.)

Thus a simple unital conformal algebra is always differential. Utilizing the classification of algebras of GKdimension 1 [33], we can finally obtain the algebraic description of Cend_n:

Theorem 3.7. [28] *A simple unital associative conformal algebra of Gelfand–Kirillov dimension 1 is isomorphic to* Cend_n.

Remark 3.8. The proof of Theorem 3.7 can be extended and yield the classification of unital semisimple conformal algebras of GKdimension 1. Namely, such an algebra always embeds into a direct sum of Cend_n and a current algebra over a semiprime algebra of zero or linear growth [30]. It would be very difficult to classify all such embeddings: for instance, there exists a prime non-current subalgebra of a current algebra of linear growth [30].

3.5. Now we turn to the representation theory of Cend_n.

Just as in the case of $\mathrm{Cur}\,\mathbb{C}$-modules, a module M over a unital conformal algebra C always splits, $M = M^0 \oplus M^1$. We need only to study the structure of M^1. As in the proof of Theorem 3.6, the action of $\widetilde{1}$ gives a certain rigidity to M^1. Namely, M^1 is filtered by the submodules annihilated by $\widetilde{1}\,ⓝ$. If C is differential, i.e. $C = \mathrm{Diff}\, B$ (with a derivation δ), the lowest non-trivial component of this filtration can be made into a B-module that completely determines the structure of M^1.

Conversely, a (unitary) B-module V gives rise to a C-module $\widetilde{V} = \mathbb{C}[\partial] \otimes V$ with the action

$$\widetilde{a}_\lambda \widetilde{v} = \sum_j \partial^j (\widetilde{\delta^j(a) v}), \quad \text{where } \widetilde{v} = 1 \otimes v, v \in V, \text{ and } a \in A. \tag{3.3}$$

Example 3.9. A finite irreducible module E_n^α over Cend_n constructed in Remark 3.1 has the form $E_n^\alpha = \widetilde{\mathbb{C}^n}$, where ∂_t acts on $\mathbb{C}^n$ as α.

This example can be generalized by considering modules $U \otimes \mathbb{C}^n$ over $\mathbb{C}[\partial_t] \otimes \mathrm{Mat}_n(\mathbb{C})$ with ∂_t acting on U as $\alpha \in \mathrm{End}(U)$. Thus we obtain a Cend_n-module $E_n^\alpha(U) = \widetilde{U \otimes \mathbb{C}^n}$.

Call a C-module M *unitary* if $M = M^1$. The discussion above implies that there is a bijection between unitary C-modules and unitary B-modules. We can make this statement more precise: consider the category $\mathrm{Rep}\, C$ whose objects are unitary C-modules and morphisms are homomorphisms that commute with $a\,ⓝ$ for every $a \in C, n \in \mathbb{Z}_{\geq 0}$. Then

Theorem 3.10. [29] $\mathrm{Rep}\, B \simeq \mathrm{Rep}\, C$.

As the above equivalence is constructed explicitly, we easily deduce that irreducibles correspond to irreducibles, indecomposables to indecomposables, and that an extension of C-modules arises from an extension of corresponding B-modules.

The only concept from representation theory that does not automatically survive is faithfulness: if $\widetilde{V}$ is faithful, we can only conclude that $\mathrm{Ann}\, V$ does not contain any nonzero δ-stable ideals.

Remark 3.11. It would be more useful to define a sort of "conformal category," i.e. to define $\mathrm{Rep}\, C$ for any conformal algebra with the morphisms also carrying a conformal structure. So far the attempts at such definition have been unsuccessful.

Corollary 3.12. *Finite irreducible* Cend_n*-modules are of the form* E_n^α*. Finite indecomposable* Cend_n*-modules are of the form* $E_n^\alpha(U)$ *for an indecomposable* $\alpha \in \mathrm{End}(U)$*.*

The above result was first stated in [20]. Another proof was given in [8]. Though more calculation-heavy, it also works for certain non-unital conformal algebras (see below).

3.6. Finite modules over gc_n look similar to Cend_n-modules; however, the classification methods here are entirely different.

We begin by constructing such modules. Since every representation of Cend_n gives rise to a representation of gc_n, we already have the family $E_n^\alpha(U)$ of gc_n-modules (see Example 3.9).

Recall that we can also construct contragradient modules $E_n^\alpha(U)^*$ (see Remark 2.2).

Theorem 3.13. [20, 7] *A finite irreducible* gc_n*-module has the form* E_n^α *or* $(E_n^\alpha)^*$*.*

The first step of the proof is to look at certain representations of the coefficient algebra instead. In fact, for a conformal algebra C it suffices to consider modules over the *annihilation algebra* spanned by ∂ and coefficients $a(n)$, $n \geq 0$. Indeed, a C-module M can be viewed as a module over the annihilation algebra. Conversely, a module V over the annihilation algebra such that for each $v \in V$, $a(n)v = 0$ for $n \gg 0$, gives rise to a C-module.

The annihilation algebra of gc_n is the direct sum of a one-dimensional Lie algebra and the algebra of matrices of regular differential operators on the line. Since we are interested in finite gc_n-modules here, we need to consider only modules with finite-dimensional graded components (i.e. *quasifinite*). Therefore, the proof of Theorem 3.13 comes down to classifying quasifinite modules over the Lie algebra of regular differential operators on the line. This was achieved in [7] via the technique developed in [21].

It can be shown that modules over the annihilation algebra of gc_n are completely reducible [22]. Thus all finite indecomposable modules over gc_n are of the form $E_n^\alpha(U)$ or $E_n^\alpha(U)^*$.

3.7. We have already discussed the finite subalgebras of Cend_n and gc_n. Here we present a continuous family of infinite subalgebras of Cend_n acting irreducibly on $\mathbb{C}[\partial]^n$.

Let $P(\partial_t)$ be a matrix in $\mathrm{Mat}_n(\mathcal{D}(\mathbb{C}^\times))$. The formal distributions from Cend_n whose coefficients lie in $\mathrm{Mat}_n(\mathcal{D}(\mathbb{C}^\times))P(\partial_t)$ form a subalgebra (in fact, a left ideal) of Cend_n denoted $\mathrm{Cend}_{n,P}$. When $P(\partial_t)$ is non-degenerate, the conformal algebra $\mathrm{Cend}_{n,P}$ acts irreducibly on $\mathbb{C}[\partial]^n$.

By applying first elementary transformations to P and then conjugating $\mathrm{Cend}_{n,P}$, we arrive at an isomorphic subalgebra for $P = \mathrm{diag}(p_1(\partial_t), \dots, p_n(\partial_t))$, where $p_i(\partial_t) \neq 0$ are monic and $p_i | p_{i+1}$. Such p_i's are called elementary divisors. Therefore, we obtain a family of non-isomorphic subalgebras of Cend_n that act irreducibly on $\mathbb{C}[\partial]$ and is parametrized by sequences of elementary divisors.

Conjecture 3.14. [20][1] *A subalgebra of* Cend_n *that acts irreducibly on the standard module* $\mathbb{C}[\partial]^n$ *is conjugate to either* $\mathrm{Cur}\,\mathrm{Mat}_n(\mathbb{C})$ *or* $\mathrm{Cend}_{n,P}$ *for a non-degenerate* P*.*

That all such finite subalgebras are conjugate to $\mathrm{Cur}\,\mathrm{Mat}_n(\mathbb{C})$ can be deduced from the conformal Cartan–Jacobson theorem [11].

Another particular case of the conjecture is its restriction to unital subalgebras; here the result follows from Theorem 3.7.

[1] *Added in proof.* Recently P. Kolesnikov has announced a proof of this conjecture; details are forthcoming.

It is possible to relax the definition of unitality and consider associative conformal algebras that contain $\mathrm{Cur}\,\mathbb{C}$ (in this case $\widetilde{1}$ is called a *conformal idempotent*). For such algebras, the conjecture also holds [36].

However, for arbitrary subalgebras, the only settled case is $n = 1$. Moreover, here one can classify all subalgebras of Cend_1 by carefully investigating subalgebra elements of minimal degree (with respect to both bases $\{J^m\}$ and $\{\widetilde{f(\partial_t)}\}$), similar to the classical proof that the algebra $\mathbb{C}[x]$ is principal [8].

3.8. One of the crucial observations of [8] is that for an associative algebra C and an irreducible C-module M, $M \simeq C_{-\partial-\lambda}m|_{\lambda=\alpha}$ for some $m \in M, \alpha \in \mathbb{C}$. Thus we obtain a surjective map from the set of maximal left ideals of C to the set of non-trivial irreducible C-modules (taken up to isomorphism). Hence,

Theorem 3.15. [8] *A finite irreducible* $\mathrm{Cend}_{n,P}$*-module is isomorphic to* E_n^α.

However, the category of representations of $\mathrm{Cend}_{n,P}$ is very different from that of Cend_n; it is actually wild for P of high degree [16]. This can be seen by constructing the ext-quiver (adapted to conformal algebras) for the finite-dimensional extensions of irreducible $\mathrm{Cend}_{n,P}$-modules classified in [8].

Remark 3.16. If one could define a conformal category (i.e. a category of conformal objects, where morphism carry a conformal structure as well, see Remark 3.11), then perhaps representations of $\mathrm{Cend}_{n,P}$ could be described as a deformation of $\mathrm{Rep}\,\mathrm{Cend}_n$. Moreover, an analog of the density theorem for endomorphisms in such a category might help in solving Conjecture 3.14.

3.9. Above we passed from Cend_n to its subalgebras $\mathrm{Cend}_{n,P}$. In the Lie case, we go from gc_n to $\mathrm{gc}_{n,P}$ (either by introducing the Lie bracket on $\mathrm{Cend}_{n,P}$ or by also considering the subalgebra of formal distributions of gc_n with coefficients divisible on the right by P). Thus, we obtain subalgebras of gc_n acting irreducibly on $\mathbb{C}[\partial]^n$.

But there is more. Since gc_n is simple, it can be viewed as both the analog of gl_n and sl_n. What about analogs of other simple Lie algebras? In particular, what are orthogonal and symplectic conformal Lie algebras?

Consider an anti-involution * on $\mathcal{D}(\mathbb{C}^\times)$: $\partial_t^* = -\partial_t, t^* = t$. It can be extended to $\mathrm{Mat}_n(\mathbb{C}) \otimes \mathcal{D}(\mathbb{C}^\times)$ by applying * to the second component and a matrix anti-involution to the first. This gives us an anti-involution of Cend_n.

Theorem 3.17. [8] *Up to conjugation, all anti-involutions of* Cend_n *are of this form.*

We continue mimicking the constructions of orthogonal and symplectic Lie algebras. Let σ be an anti-involution on Cend_n that arises from a symmetric (resp. skew-symmetric) involution of $\mathrm{Mat}_n(\mathbb{C})$. The fixed points of $-\sigma$ form the *orthogonal conformal algebra* oc_n (resp. *symplectic conformal algebra* spc_n). As gc_n, both oc_n and spc_n are simple.

We can go further and define orthogonal and symplectic subalgebras $\mathrm{oc}_{n,P}$ and $\mathrm{spc}_{n,P}$ of $\mathrm{gc}_{n,P}$. However, this can be done only for hermitian and anti-hermitian P's respectively.

Remark 3.18. There exists another construction of oc_n and spc_n that is more representation-theoretic in spirit. Here we begin by defining a *conformal form* on a $\mathbb{C}[\partial]$-module V: $\langle\, ,\, \rangle_\lambda : V \otimes V \to \mathbb{C}[\lambda]$. The form is *bilinear* if

$$\langle \partial v, w\rangle_\lambda = -\lambda\langle v, w\rangle_\lambda = -\langle v, \partial w\rangle_\lambda, \quad v, w \in V$$

and *symmetric* (resp. *skew-symmetric*) if $\langle v, w\rangle_\lambda = \langle w, v\rangle_{-\lambda}$ (resp. $-\langle w, v\rangle_{-\lambda}$).

If V is free over $\mathbb{C}[\partial]$, the form is completely determined by its action on the basis $\{e_i\}$, i.e. by a matrix $P(\lambda)$ with entries $\langle e_i, e_j\rangle_\lambda$.

A conformal bilinear form gives rise to a map $V \to V^*$, which establishes an isomorphism between V and $P(-\partial)V^*$ (for a non-degenerate P).

Thus, though we can not define adjoints with respect to a given form for all operators in Cend_n, there is a well-defined adjoint for elements of $\mathrm{Cend}_{n,P}$. (The details here are rather involved, see [8].) Elements that are skew-symmetric with respect to taking the adjoint form up the Lie conformal algebra $\mathrm{oc}_{n,P}$ (if the corresponding form is symmetric) or $\mathrm{spc}_{n,P}$ (resp. skew-symmetric).

As a companion to Conjecture 3.14, we have

Conjecture 3.19. [35, 8] *An infinite subalgebra of* gc_n *that acts irreducibly on the standard module* $\mathbb{C}[\partial]^n$ *is conjugate to either* $\mathrm{gc}_{n,P}$, $\mathrm{oc}_{n,P}$, *or* $\mathrm{spc}_{n,P}$ *for a non-degenerate* P *(if defined).*

Remark 3.20. The classification of finite subalgebras of gc_n acting irreducibly on $\mathbb{C}[\partial]^n$ is contained in [11].

Remark 3.21. It is worth to emphasize that here we get only the analogs of classical series A_n, B_n, C_n, and D_n (though the series are not discrete). There is no place for exceptional Lie conformal algebras in this conjecture.

And indeed, so far all attempts to construct such by the analogs of ordinary methods (Tits–Kantor–Koecher construction, sums of representations of algebras of small rank, etc.) have failed.

The existing evidence for this conjecture covers several important cases. In all cases we assume a presence of an important finite subalgebra:

- [36] If C is a simple Lie conformal algebra that has a faithful finite representation and such that $C \supset \mathrm{Cur}\,\mathrm{sl}_2$, then C is one of the subalgebras from Conjecture 3.19.
- [12] If C is a subalgebra of gc_n that acts irreducibly on $\mathbb{C}[\partial]^n$ and is fixed by the action of $L_{(n)}$, $n = 0, 1, 2$, for the Virasoro element $L = \widetilde{\partial}_t + \alpha\partial\widetilde{1}$, then C is one of the subalgebras from Conjecture 3.19.

Remark 3.22. The second statement above can be reformulated: the (centerless) Virasoro Lie algebra contains an sl_2 spanned by L_{-1}, L_0, and L_1. A subalgebra fixed by this sl_2 and acting irreducibly on $\mathbb{C}[\partial]$ is from Conjecture 3.19.

Finally, as far as the study of representations of infinite subalgebras of gc_n goes, only few inroads have been made: [7] has the classification of finite representations of subalgebras of gc_1 containing a Virasoro subalgebra but this is all.

4. Future Developments

4.1. We restate here Conjectures 3.14 and 3.19:

Conjecture 4.1. *Let* C *be an infinite conformal algebra that acts faithfully and irreducibly on* $\mathbb{C}[\partial]$*. Then*

- *if* C *is associative, it is isomorphic to* $\mathrm{Cend}_{n,P}$*;*
- *if* C *is Lie, it is isomorphic to either* $\mathrm{gc}_{n,P}$, $\mathrm{oc}_{n,P}$, *or* $\mathrm{spc}_{n,P}$.

It is well known that any study of infinite-dimensional Lie algebras of linear growth is in general hopeless; fortunately, there is a natural subclass of affine algebras that gives us a controllable rich theory. Since these algebras are finite from the conformal point of view, the subalgebras of gc_n (or, rather, their coefficients) form a good class of algebras of quadratic growth.

More generally, one should study finite subalgebras of gc_n as well: this may lead, for instance, to the classification of Virasoro elements (see Remark 3.3) and other important results.

In view of Theorem 3.7 and its generalization in [36], we also propose a generalization of Conjecture 3.14:

Conjecture 4.2. *Let C be a simple associative algebra of Gelfand–Kirillov dimension* 1*. Then C is isomorphic to* $\mathrm{Cend}_{n,P}$ *for some n and P.*

4.2. A closely related issue is the study of representations of $\mathrm{Cend}_{n,P}$, $\mathrm{gc}_{n,P}$, $\mathrm{oc}_{n,P}$ and $\mathrm{spc}_{n,P}$.

We have already mentioned in Remarks 3.11 and 3.16 that the best strategy here might be to define a conformal category with the space of morphisms carrying a conformal structure and then describe this category in terms of some equivalent data. Though successful for unital conformal algebras (see Theorem 3.10), this task is very hard in general. This is not the only obstacle for such a project. For instance, as mentioned above, there is no bijection between the set of maximal left ideals and irreducible modules–i.e. even if one could define a "conformal kernel," it would yield less information than its ordinary analog.

In the Lie case, these questions are closely related to the study of representations of $\widehat{\mathrm{gl}_\infty}$ and its subalgebras. This is a very important subject for infinite-dimensional representation theory and hopefully, the use of conformal language can move it further.

4.3. Representations of Lie algebras are a source for a lot of combinatorics. What about their conformal counterparts?

The study of subalgebras of gc_n that are normalized by the sl_2 part of a Virasoro element (see Remark 3.22) produced a surprising connection with classical Jacobi polynomials [12]. This seems to be the only deep combinatorial result in the conformal algebra field but we can hope for more: for instance, character formulas for gc_n (and, for that matter a good definition of "conformal" characters) and its subalgebras should turn out very interesting from the combinatorial point of view.

4.4. Finally, we should mention a generalization of conformal algebras.

It is clear that instead of $\mathbb{C}[\partial]$-modules in Definition 1.1 we can consider modules over $\mathbb{C}[\partial_1, \dots, \partial_k]$ (and take n in ⓝ to be a multiindex). A more involved procedure allow us to endow modules over any cocommutative Hopf algebra with a conformal-like structure. Such objects are called *pseudoalgebras*.

So far we have the classification of finite pseudoalgebras, the beginnings of representation theory (both in [1]), and the theory of unital pseudoalgebras [29]. It should be mentioned that simple finite Lie pseudoalgebras arise from either affine algebras or algebras of Cartan type (of any GK dimension), so in a sense this theory unifies the Kac–Moody and the Cartan type sides of infinite-dimensional Lie algebras, cf. Remark 1.12. As far as infinite pseudoalgebras are concerned, some results in [29] (for the associative case only) suggest that their theory is rich and manageable. These objects should be given more attention.

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