Applied Algebra, MAT312/AMS351
Practice Problems for Midterm II

(1) Let $R = \{(a,b) \mid a \equiv b \mod 5\}$ be a subset of $\mathbb{Z} \times \mathbb{Z}$. Prove or disprove that aRb is an equivalence relation on $\mathbb{Z}$.

(2) Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 2 & 7 & 3 & 1 & 5 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 5 & 1 & 4 & 2 & 7 \end{pmatrix}$. Compute $\pi\sigma$, π^{-1}. Determine orders and signs of π and σ.

(3) Prove that for any permutation π, the permutation $\pi^{-1}(12)\pi$ is a transposition.

(4) Leat a, b be elements of a group G. Solve equations $a^{-1}x = b$ and $xa^{-1}b = e$.

(5) Let G be a group such that for any two elements a, b in G, $(ab)^2 = a^2b^2$. Prove that G is abelian.

(6) Let G be a group. Define the relation of *conjugacy* on G: aRb if and only if there exists $g \in G$ such that $b = g^{-1}ag$. Prove that this is an equivalence relation.

(7) Compute orders of the following elements of the group $(\mathbb{C}^\times, \cdot)$: $3i$, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

(8) For a matrix A denote its transpose by A^t. A is orthogonal if $A^{-1} = A^t$ (A^t means the transpose of A). Prove that the set of invertible orthogonal $n \times n$ matrices is a subgroup of $GL(n, \mathbb{R})$. (*Hints:* First recall – or deduce – that $(AB)^t = B^tA^t$ and $(A^{-1})^t = (A^t)^{-1}$.)

(9) Let R be a commutative ring such that $1 + 1 = 0$. Prove that for any $x, y \in R$, $(x+y)^2 = x^2 + y^2$.

(10) Prove that the subset $\{a + bj | a, b \in \mathbb{R}\}$ of $\mathbb{H}$ is a field.