Applied Algebra, MAT312/AMS351
Practice Problems for Midterm 1

1. Find the greatest common divisor of $12n + 1$ and $30n + 2$.

2. Prove that for every natural number n, the number $3^{2n+2} + 8n - 9$ is divisible by 16.

3. Recall that the Fibonacci sequence is defined as $F_1 = 1, F_2 = 1$, and then for every $n > 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that for every n, $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$.

4. Find all $n > 2$ such that $n^3 - 3$ is divisible by $n - 1$.

5. When questioned by the police, the suspect claimed that he did not remember his home address but could definitely recall that the house number is less than 1000 and is divisible by 7, 11, and 13. Is the suspect telling the truth?
   And what if he said that the number was divisible by 7, 11, and 14?

6. Let a, b, and c be positive integers such that $a^2 + b^2 = c^2$. Prove that at least one of them is divisible by 3.
   (*Hint*: reduce mod 3.)

7. Solve the following linear congruences:
   (a) $5x \equiv 7 \mod 31$;
   (b) $2x \equiv 19 \mod 2006$;
   (c) $19x + 3 \equiv 4 \mod 83$.

8. Find the minimal positive integer satisfying the following conditions:
   (i) when divided by 7, its remainder is 4,
   (ii) when divided by 12, its remainder is 5.

9. Compute $\phi(1001)$, $\phi(96)$.

10. Find the last two digits of 1221^{122}.