Calculus IV with Applications MAT303
Solutions to Practice Problems for Midterm II

3.1, 26. If dependent, then $f = cg$ for a constant c, i.e. $2\cos x + 3\sin x = c(3\cos x - 2\sin x)$. Then comparing coefficients at $\cos x$ and $\sin x$, we get $2 = 3c$ and $3 = -2c$ at the same time, which is impossible. Therefore, f and g are linearly independent. (Another solution: compute the Wronskian.)

3.1, 38. Char. eq-n: $4r^2 + 8r + 3 = 0$. Solutions to char eq-n: $r = -3/2, -1/2$. General sol-n: $c_1e^{-\frac{3}{2}x} + c_2e^{-\frac{1}{2}x}$.

3.2, 10. $W(f,g,h) = \begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & -2x^{-3}\ln x + x^{-3} \\ e^x & 6x^{-4} & 6x^{-4}\ln x - 5x^{-4} \end{vmatrix} =$ $e^x(-2x^{-3}(6x^{-4}\ln x - 5x^{-4}) - (-2x^{-3}\ln x + x^{-3})6x^{-4}) - e^x(x^{-2}(6x^{-4}\ln x - 5x^{-4}) - x^{-2}\ln x 6x^{-4}) + e^x(x^{-2}(-2x^{-3}\ln x + x^{-3}) - x^{-2}\ln x(-2x^{-3})) = e^{-x}(4x^{-7} + 5x^{-6} + x^{-5}) \neq 0$.

3.2, 15. General solution: $y(x) = c_1e^x + c_2xe^x + c_3x^2e^x$. Then $y'(x) = (c_1 + c_2)e^x + (c_2 + 2c_3)xe^x + c_3x^2e^x$ and $y''(x) = (c_1 + 2c_2 + 2c_3)e^x + (c_2 + 4c_3)xe^x + c_3x^2e^x$. Then we have

$$\begin{cases} 2 = y(0) = c_1 \\ 0 = y'(0) = c_1 + c_2 \\ 0 = y''(0) = c_1 + 2c_2 + 2c_3 \end{cases}$$

Hence, $c_1 = 2$, $c_2 = -2$, $c_3 = 1$. Solution: $2e^x - 2xe^x + x^2e^x$.

3.2, 23. General solution: $y(x) = y_c + y_p = c_1e^{-x} + c_2e^{3x} - 2$. Then $y'(x) = -c_1e^{-x} + 3c_2e^{3x}$. Then we get $3 = y(0) = c_1 + c_2 - 2$ and $11 = y'(0) = -c_1 + 3c_2$. Then $c_1 = 1$, $c_2 = 4$. Solution: $e^{-x} + 4e^{3x} - 2$.

3.3, 12. Char eq-n: $r^4 - 3r^3 + 3r^2 - r = 0$ or, equivalently, $r(r^3 - 3r^2 + 3r - 1) = 0$ or, equivalently, $r(r-1)^3 = 0$. Solutions to char eq-n: r=0, r=1 (with multiplicity 3). General sol-n: $c_1e^{0x} + c_2e^x + c_3xe^x + c_4x^2e^x = c_1 + c_2e^x + c_3xe^x + c_4x^2e^x$.

3.3, 16. Char eq-n: $r^4 + 18r^2 + 81 = 0$. Put $r^2 = s$, then $s^2 + 18s + 81 = 0$ and $s = -9$, thus $r = \pm\sqrt{9} = \pm 3i$ (each root with multiplicity 2). Alternative approach: rewrite equation as $(r^2 + 9)^2 = 0$, then as $(r + 3i)^2(r - 3i)^2 = 0$. General sol-n: $c_1\cos 3x + c_2x\cos 3x + c_3\sin 3x + c_4x\sin 3x$.

3.4, 16. $3x'' + 30x' + 63x = 0$. The characteristic equation is $3r^2 + 30r + 63 = 0$, thus $r = \dfrac{-30 \pm \sqrt{30^2 - 4\cdot 63\cdot 3}}{2\cdot 3} = -7, -3$. The roots are real and distinct, therefore the system is overdamped. General solution: $x(t) = c_1e^{-3t} + c_2e^{-7t}$. Then $v(t) = x'(t) = -3c_1e^{-3t} - 7c_2e^{-7t}$. From $x(0) = 2, v(0) = 2$, we have $c_1 + c_2 = 2, -3c_1 - 7c_2 = 2$. Thus $c_1 = 4, c_2 = -2$. Position function: $4e^{-3t} - 2e^{-7t}$.

In the undamped case, the equation is $3x'' + 63x = 0$. Then $\omega_0 = \sqrt{63/3} = \sqrt{21}$. General solution: $x(t) = C\cos(\omega_0 t - \alpha) = C\cos(\sqrt{21}t - \alpha)$. Then $v(t) = x'(t) = -\sqrt{21}C\sin(\sqrt{21}t - \alpha)$. From $x(0) = 2, v(0) = 2$, we have $2 = C\cos(-\alpha)$ and $2 = -\sqrt{21}C\sin(-\alpha)$. $C = 2/\cos(-\alpha)$, thus $2 = -2\sqrt{21}\sin(-\alpha)/\cos(-\alpha) = -2\sqrt{21}\tan(-\alpha)$. It follows that $\alpha = \arctan(1/\sqrt{21})$. If $\tan\alpha = 1/\sqrt{21}$, then $\cos\alpha = \sqrt{21/22}$. Hence $C = 2\sqrt{22/21}$.

3.4, 20. $2x'' + 16x' + 40x = 0$. The characteristic equation is $2r^2 + 16r + 40 = 0$, thus $r = \dfrac{-16 \pm \sqrt{16^2 - 4\cdot 40\cdot 2}}{2\cdot 2} = -4 \pm 2i$. The roots are complex, therefore the system is underdamped. General solution $x(t) = Ce^{-4t}\cos(2t - \alpha)$ (i.e. $p = 4, \omega_1 = 2$). $x'(t) = -4Ce^{-4t}\cos 2t - \alpha - 2Ce^{-4t}\sin(2t - \alpha)$. From

$x(0) = 5, x'(0) = 4$, we have $5 = C\cos(-\alpha)$, $4 = -4C\cos(-\alpha) - 2C\sin(-\alpha)$. $C = 5/\cos(-\alpha)$ and $-12 = 5\sin(-\alpha)/\cos(-\alpha)$. Thus $\tan(-\alpha) = -12/5$, $\tan(\alpha) = 12/5$. Thus $\alpha = \arctan(12/5)$ and $\cos(\alpha) = 5/13$, $C = 13$. Position function: $x(t) = 13\cos(2t - \arctan(12/5)$.

3.5, 14. Associated homogeneous equation: $y^{(4)} - 2y'' + y = 0$. Characteristic equation: $r^4 - 2r^2 + 1 = 0$ or, equivalently, $(r^2-1)^2 = 0$ or, equivalently, $(r-1)^2(r+1)^2 = 0$. Solutions: $r = \pm 1$ (each with multiplicity 2). General solution: $y_c = c_1e^x + c_2xe^x + c_3e^{-x} + c_4xe^{-x}$.

Since $f(x) = xe^x$, $f'(x) = xe^x + e^x$. A linear combination of $f(x)$ and its derivatives has the form $Ae^x + Bxe^x$. Trial solution: $x^s(Ae^x + Bxe^x)$. Both e^x and xe^x are particular solutions of the associated equation, hence we take $s = 2$. Trial solution: $Ax^2e^x + Bx^3e^x$.

Plug in trial solution: $(Ax^2e^x + Bx^3e^x)^{(4)} - 2(Ax^2e^x + Bx^3e^x)'' + Ax^2e^x + Bx^3e^x = xe^x$.
$A(12e^x + 8xe^x + x^2e^x) + B(24e^x + 36xe^x + 12x^2e^x + x^3e^x) - 2A(2e^x + 4xe^x + x^2e^x) - 2B(6xe^x + 6x^2e^x + x^3e^x)Ax^2e^x + Bx^3e^x = xe^x$.
$(12A + 24B - 4A)e^x + (8A + 36B - 8A - 12B)xe^x + (A + 12B - 2A - 12B + A)x^2e^x + (B - 2B + B)x^3e^x = xe^x$.
Therefore, $8A + 24B = 0$ and $24B = 1$. It follows that $B = 1/24, A = -1/8$, and $y_p = -\frac{1}{8}x^2e^x + \frac{1}{24}x^3e^x$.

3.5, 17. Associated homogeneous equation: $y'' + y = 0$. Characteristic equation: $r^2 + 1 = 0$. Solutions: $r = \pm i$. General solution: $y_c = c_1\cos x + c_2\sin x$.

Since $f(x) = \sin x + x\cos x$, we consider $\sin x$ and $x\cos x$ separately. Derivatives of $\sin x$ are $\pm\cos x$ or $\pm\sin x$, thus the trial solution for $\sin x$ has form $x^s(A\sin x + B\cos x)$. Both $\sin x$ and $\cos x$ are solutions of the associated equations, thus we must take $s = 1$. For $x\cos x$, the linear combinations of all its derivatives will have the form $Cx\sin x + Dx\cos x + E\sin x + F\cos x$. Thus the trial solution here is $x^s(Cx\sin x + Dx\cos x + E\sin x + F\cos x)$. Again, because both $\sin x$ and $\cos x$ are solutions of the associated equations, thus we must take $s = 1$. The sum of both trial solutions gives us the trial solution for $f(x)$: $x(A\sin x + B\cos x) + x(Cx\sin x + Dx\cos x + E\sin x + F\cos x)$. Combining similar terms together (and relabeling undetermined coefficients), we get $ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x$.

Plug in trial solition: $(ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x)'' + ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x = \sin x + x\cos x$.
$((2c - 2b)\sin x + (2a + 2d)\cos x + (-2a - 4d)x\sin x + (-2b + 4c)x\cos x - cx^2\sin x - dx^2\cos x) + ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x = \sin x + x\cos x$.
$(2c - 2b)\sin x + (2a + 2d)\cos x + (-2a - 4d + a)x\sin x + (-2b + 4c + b)x\cos x = \sin x + x\cos x$.
Therefore, $2c - 2b = 1$, $2a + 2d = 0$, $-a - 4d = 0$, $-b + 4c = 1$. It follows that $a = d = 0$, $b = -1/3$, $c = 1/6$.

3.5, 38. Associated homogeneous equation: $y'' + 2y' + 2y = 0$. Characteristic equation: $r^2 + 2r + 2 = 0$. Solutions: $r = -1 \pm i$. General solution: $y_c = c_1e^{-x}\cos x + c_2e^{-x}\sin x$.

Since $f(x) = \sin 3x$, $f'(x) = 3\cos 3x$. A linear combination of $f(x)$ and its derivatives has the form $A\sin 3x + B\cos 3x$. Trial solution: $x^s(A\sin 3x + B\cos 3x)$. Neither $\sin 3x$ nor $\cos 3x$ are particular solutions of the associated equation, hence we take $s = 0$. Trial solution: $A\sin 3x + B\cos 3x$.

Plug in trial solution: $(A\sin 3x+B\cos 3x)''+2(A\sin 3x+B\cos 3x)'+2(A\sin 3x+B\cos 3x)=\sin 3x$.
$-9A\sin 3x-9B\cos 3x+2(3A\cos 3x-3B\sin 3x)+2(A\sin 3x+B\cos 3x)=\sin 3x$.
$(-9A-6B+2A)\sin 3x+(-9B+6A+2B)\cos 3x=\sin 3x$.
Hence $-7A-6B=1$ and $-7B+6A=0$. Thus $A=-7/85$, $B=-6/85$, and $y_p=-\frac{7}{85}\sin 3x-\frac{6}{85}\cos 3x$.

The general solution is $y(x)=c_1e^{-x}\cos x+c_2e^{-x}\sin x-\frac{7}{85}\sin 3x-\frac{6}{85}\cos 3x$. Thus $y'(x)=-c_1e^{-x}\cos x-c_1e^{-x}\sin x-c_2e^{-x}\sin x+c_2e^{-x}\cos x-\frac{21}{85}\cos 3x+\frac{18}{85}\sin 3x=(c_2-c_1)e^{-x}\cos x+(-c_1-c_2)e^{-x}\sin x-\frac{21}{85}\cos 3x+\frac{18}{85}\sin 3x$.

$2=y(0)=c_1-\frac{6}{85}$ and $0=y'(0)=(c_2-c_1)-\frac{21}{85}$. Then $c_1=\frac{176}{85}$ and $c_2=\frac{197}{85}$. Solution: $y(x)=(176e^{-x}\cos x+197e^{-x}\sin x-7\sin 3x-6\cos 3x)/85$.

4.1, 6. Set $z=x'$, $w=y'$. Then $x''=z'$ and $y''=w$. Answer:

$$\begin{cases} z'-5x+4y=0\\ w'+4x-5y=0\\ z=x'\\ w=y' \end{cases}$$

4.1, 17. $y=x'$, thus $y'=x''$. From $y'=6x-y$, $x''=6x-x'$, i.e. $x''+x'-6x=0$. Char eq-n: $r^2+r-6=0$. $r=-3,2$. General solution: $x(t)=c_1e^{-3t}+c_2e^{2t}$, $y(t)=x'(t)=-3c_1e^{-3t}+2c_2e^{2t}$.

$1=x(0)=c_1+c_2$, $2=y(0)=-3c_1+2c_2$. Hence $c_1=0, c_2=1$. Solution: $x(t)=e^{2t}, y(t)=2e^{2t}$.

4.2, 4. $y=3x-x'$, hence the second eq-n becomes $(3x-x')'=5x-3(3x-x')$. Then $x''-4x=0$. Char. eq-n: $r^2-4=0$. $r=\pm 2$, thus $x(t)=c_1e^{2t}+c_2e^{-2t}$ and $y(t)=3x-x'=c_1e^{2t}+5c_2e^{-2t}$.

$1=x(0)=c_1+c_2, -1=y(0)=c_1+5c_2$. Hence $c_1=3/2, c_2=-1/2$. Solution: $x(t)=\frac{3}{2}e^{t}-\frac{1}{2}e^{-t}, y(t)=\frac{3}{2}e^{t}-\frac{5}{2}e^{-t}$.