Calculus IV with Applications MAT303
Solutions to Practice Problems for Midterm 1

1.3, 17. $f(x,y) = \frac{x-1}{y}$ and $D_y f(x,y) = -\frac{x-1}{y^2}$ are continuous in the neighborhood of $(0,1)$, thus the solution near $x = 0$ exists and is unique.

1.3, 18. $f(x,y) = \frac{x-1}{y}$ is not continuous at $(1,0)$, thus neither existence nor uniqueness are guaranteed.

1.4, 13. Separable equation: $\frac{y^3}{y^4+1}dy = \cos x dx$. $\int \frac{y^3}{y^4+1}dy = \frac{1}{4}\ln(y^4+1)$ (after substitution $u = y^4+1$), thus $\frac{1}{4}\ln(y^4+1) = \sin x + C_1$ or $\ln(y^4+1) = 4\sin x + C_2$ (replace $4C_1$ with C_2). Then $y = (Ce^{\sin x} - 1)^{1/4}$, $C > 0$.

1.4, 25. Separable equation: $xy' = (2x^2+1)y$, so $\frac{dy}{y} = \frac{2x^2+1}{x}dx$. $\int 2x + \frac{1}{x}dx = x^2 + \ln|x| + C_1$, hence after integrating both sides $\ln|y| = x^2 + \ln|x| + C_1$. Thus $y = Ce^{x^2}|x|$.

The initial condition $y(1) = 1$ gives $C = e^{-1}$, hence $y = xe^{x^2-1}$.

1.5, 13. Integrating factor $\rho(x) = e^{\int 1 dx} = e^x$. Thus $(e^x y)' = e^x e^x$ and $e^x y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$. Dividing both sides by e^x, we get $y = \frac{1}{2}e^x + Ce^{-x}$.

The initial conditions $y(0) = 1$ gives $C = \frac{1}{2}$, hence $y = \frac{1}{2}(e^x + e^{-x})$.

1.5, 23. $y' + \frac{2x-3}{x}y = 4x^3$. Integrating factor: $\exp\left(\int 2 - \frac{3}{x}dx\right) = \exp(2x - 3\ln x) = e^{2x}/x^3$. Thus $\left(\frac{e^{2x}}{x^3}y\right)' = 4e^{2x}$ and $\frac{e^{2x}}{x^3}y = 2e^{2x} + C$. Finally, $y = x^3(2 + Ce^{-2x})$.

1.6, 7. $y' = \frac{x^2}{y^2} + \frac{y}{x}$. This is a homogeneous equation and substitution $v = \frac{y}{x}$ gives a separable equation $xv' = v^{-2}$. Thus $v^2 dv = \frac{dx}{x}$ and $\frac{v^3}{3} = \ln|x| + C$. Substituting back, we find that $y^3 = 3x^3(\ln|x| + C)$ or $y = x(3(\ln|x| + C))^{1/3}$.

1.6, 28. In this equation y is present only in the expression e^y, so we make the substitution $v = e^y$. Then $v' = e^y y'$ and the equation becomes $xv' = 2(v + x^3 e^{2x})$. Rewrite as: $v' - \frac{2}{x}v = 2x^2 e^{2x}$. Integrating factor: $e^{\int -2/x dx} = x^{-2}$. Thus $(x^{-2}v)' = 2e^{2x}$, i.e. $x^{-2}v = \int e^{2x}dx = e^{2x} + C$. Then $v = (e^{2x} + C)x^2$ or $e^y = (e^{2x} + C)x^2$. Finally, $y = \ln(e^{2x}x^2 + Cx^2)$.

1.6, 39. Checking that the equation is exact: $\frac{\partial}{\partial y}(3x^2 y^3 + y^4) = 9x^2 y^2 + 4y^3$, $\frac{\partial}{\partial x}(3x^3 y^2 + y^4 + 4xy^3) = 9x^2 y^2 + 4y^3$.

Solution: $F(x,y) = \int 3x^2 y^3 + y^4 dx = x^3 y^3 + xy^4 + C(y)$. To determine $C(y)$, $3x^3 y^2 + y^4 + 4xy^3 = \frac{\partial}{\partial y}F(x,y) = 3x^3 y^2 + 4xy^3 + C'(y)$. Hence $C'(y) = y^4$ and $C(y) = \frac{y^5}{5} + C_1$. Answer: $F(x,y) = x^3 y^3 + xy^4 + \frac{y^5}{5} + C_1 = 0$ or $5x^3 y^3 + 5xy^4 + y^5 + C = 0$.

1.6, 48. Substitution: $y' = p(x)$. Then $x^2 p' + 3xp = 2$ or $p' + \frac{3}{x}p = 2x^{-2}$. Integrating factor: $e^{\int 3/x dx} = x^3$. The equation becomes $(x^3 p)' = 2x$, i.e. $x^3 p =$

$x^2 + C$ or $p = x^{-1} + Cx^{-3}$. Integrating p to obtain y, we get $y = \ln x + Ax^{-2} + B$ ($A = -C/2$).

1.6, 49. Substitution: $y' = p(y)$. Then $y'' = \frac{dp}{dy}y' = \frac{dp}{dy}p$ and the equation becomes $yp'p + p^2 = yp$. Then $p' + \dfrac{p}{y} = 1$. Integrating factor: $e^{\int 1/y dy} = y$. Then $(yp)' = y$. Hence $yp = \dfrac{y^2}{2} + C_1$ or $p = \dfrac{y^2 + C}{2y}$. (Here $C = 2C_1$.) To obtain y, we solve the differential equation $y' = \dfrac{y + C}{2y}$: $\dfrac{2ydy}{y^2 + C} = dx$, hence $\ln|y^2 + C| = x + A$ and $y = (\pm(Be^x - C))^{1/2}$.

2.2, 10. Critical points: $7x - x^2 - 10 = 0$ has solutions $x = 2, 5$. $x' < 0$ on $(-\infty; 2)$ and $(5; \infty)$; $x' > 0$ on $(2; 5)$. Therefore 2 is an unstable critical point and 5 is stable.

Solution: $\dfrac{dx}{7x - x^2 - 10} = dt$. From $7x - x^2 - 10 = -(x-2)(x-5)$, $\dfrac{1}{7x - x^2 - 10} = \dfrac{1}{3(x-2)} - \dfrac{1}{3(x-5)}$. Thus $\displaystyle\int \dfrac{dx}{7x - x^2 - 10} = \int \dfrac{1}{3(x-2)} - \dfrac{1}{3(x-5)}dx = \dfrac{1}{3}(\ln|x-2| - \ln|x-5|) = \dfrac{1}{3}\ln\left|\dfrac{x-2}{x-5}\right|$. Therefore, the solution is $\dfrac{1}{3}\ln\left|\dfrac{x-2}{x-5}\right| = t + C_1$ or $\dfrac{x-2}{x-5} = Ce^{3t}$. Solving for x, we obtain $x = \dfrac{2 - 5Ce^{3t}}{1 - Ce^{3t}}$.

2.2, 21. $kx - x^3 = x(k - x^2)$, thus the critical points are $x = 0$ and $x = \pm\sqrt{k}$ (only when $k \geq 0$).

(a) If $k < 0$, the only critical point is $x = 0$. $\frac{dx}{dt} > 0$ for $x < 0$ and $\frac{dx}{dt} < 0$ for $x > 0$, thus the critical point is stable.

(b) $\frac{dx}{dt} > 0$ on $(-\infty, -\sqrt{k}$ and $(0, \sqrt{k})$; $\frac{dx}{dt} < 0$ on $(-\sqrt{k}, 0)$ and $(\sqrt{k}, \infty)$.

2.3, 9. The velocity equation is $\frac{dv}{dt} = 5 - 0.1v$. The initial acceleration is positive but as velocity increases, acceleration declines. When acceleration is zero, velocity cannot increase further because such an increase will immediately make acceleration negative and lead to a decline in velocity. Thus the maximum is achieved when $5 - 0.1v = 0$, i.e. when $v = 50$.

Note that $v = 50$ is the stable critical point of the velocity equation. Therefore we know immediately that $v = 50$ is the terminal (limiting) velocity of the boat but this is not enough to conclude that it is also the maximal velocity.

2.3, 12. $W = 640$, $B = 62.5 \cdot 8 = 500$, and $m = 640/32 = 20$. The velocity equations is $m\dfrac{dv}{dt} = -W + B - v$ ($F_R = -1 \cdot v$ because as vectors they have opposite directions). Thus $\dfrac{mdv}{-W + B - v} = dt$. Integrating both sides, we obtain $-m\ln|W - B + v| = t + C$. Since $v(0) = 0$, $C = -m\ln(W - B)$. Therefore, $v = e^{-(t+C)/m} + W - B = e^{-t/m}(W - B) + W - B$, so finally $v = (W - B)(e^{-t/m} - 1)$. The equation of motion is $\dfrac{dy}{dt} = v$, hence $y = (W - B)(-me^{-t/m} - t) + C_1$. Drums are dropped from zero depth, i.e. $y(0) = 0$ which makes the equation $y = (W - B)(-me^{-t/m} - t + m)$.

Now we compute at what time $v = -75$ and plug the answer into the equation of motion. Answer: 648 ft.