# MAT 126: PRACTICE FOR THE FINAL

SOLUTIONS

## Chapter 6 Review Exercises

**2.** Intersection points: between $y = 0$ and $y = x^2$: at $(0, 0)$;
between $y = 1/x$ and $y = x^2$: $1/x = x^2$ implies that $x^3 = 1$, that is, $x = 1$.

The bottom of the region is the line $y = 0$; the top is made up of two curves, $y = x^2$ and $y = 1/x$. Hence the area splits into two (each having a different "top"):

$$\int_0^1 x^2\,dx + \int_1^e \frac{1}{x}dx = \left.\frac{x^2}{2}\right|_0^1 + \ln x|_1^e = \frac{1}{2} - 0 + (\ln e - \ln 1) = \frac{1}{2} + 1 = \frac{3}{2}$$

**4.** First, find the intersection points of $x = -y$, $x = y^2 + 3y$:
$-y = y^2 + 3y$
$y^2 + 4y = 0$
$y = 0, -4$
For $-4 < y < 0$, $-y > y^2 + 3y$ (check, e.g. the values at $y = -1$). So, we integrate $\int_{-4}^{0} -y - (y^2 + 3y)dy = \left.\left(-\frac{y^3}{3} - 4\frac{y^2}{2}\right)\right|_{-4}^{0} = -\left(-\frac{(-4)^3}{3} - 4\frac{(-4)^2}{2}\right) = \frac{4^3}{6} = \frac{32}{3}$

**6.** Note that the curves $y = 1 + x$ and $y = e^{-2x}$ intersect at the point $(0, 1)$ $(e^{-2*0} = 1 = 1 + 0)$.

On the interval $[0, 1]$, $1 + x > e^{-2x}$ (because $e^{-2x} < 1$).

Hence, the volume is $\int_0^1 \pi(1 + x)^2 - \pi(e^{-2x})^2dx = \pi\int_0^1 1 + 2x + x^2 - e^{-4x}dx =$
$\pi\left.\left(x + x^2 + \frac{x^3}{3} + \frac{1}{4}e^{-4x}\right)\right|_0^1 = \pi\left(1 + 1 + \frac{1}{3} + \frac{e^{-4}}{4} - \frac{1}{4}\right) = \frac{(29 - 3e^{-4})\pi}{12}$

(The antiderivative of $e^{-4x}$, i.e. $\int e^{-4x}dx$, can be computed by taking the substitution $u = -4x$, $du = -4dx$ and $dx = -(1/4)du$. Then $\int e^{-4x}dx = (-1/4)\int e^u du = -e^u/4 + C = -e^{-4x}/4 + C$.)

**8.** Points of intersection are:
$x^3 = 2x - x^2$
$x^3 + x^2 - 2x = 0$
$x(x^2 + x - 2) = 0$
$x = 0, 1, -2$
Since the region is taken in the first quadrant, we consider only the region between $x = 0$ and $x = 1$.

Also, note that in this region $x^3 < 2x - x^2$

(a) $\displaystyle\int_0^1 2x - x^2 - x^3\, dx = \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$

(b) $\displaystyle\int_0^1 \pi(2x - x^2)^2 - \pi(x^3)^2 dx = \pi \int_0^1 4x^2 - 4x^3 + x^4 - x^6 dx = \pi\left(\frac{4x^3}{3} - x^4 + \frac{x^5}{5} - \frac{x^7}{7}\right)\Big|_0^1 =$
$\displaystyle\pi\left(\frac{4}{3} - 1 + \frac{1}{5} - \frac{1}{7}\right) = \frac{41\pi}{105}$

**9.** Points of intersection: $x = x^2$, hence $x = 0, 1$.

(a) $\displaystyle\int_0^1 \pi(x)^2 - \pi(x^2)^2 dx = \pi \int_0^1 x^2 - x^4 dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

(b) If $y = x^2$, then $x = \sqrt{y}$. Also note that for $0 \le y \le 1$, $\sqrt{y} \ge y$.

$\displaystyle\int_0^1 \pi(\sqrt{y})^2 - \pi(y)^2 dy = \pi \int_0^1 y - y^2 dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

(c) $\displaystyle\int_0^1 \pi(x^2 - 2)^2 - \pi(x - 2)^2 dx = \pi \int_0^1 x^4 - 4x^2 + 4 - x^2 + 4x - 4\, dx =$
$\displaystyle\pi \int_0^1 x^4 - 5x^2 + 4x\, dx = \pi\left(\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2\right)\Big|_0^1 = \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \frac{8\pi}{15}$

**21.** The area of a triangle with sides $a$ and $b$ and an angle $\theta$ between them is $\frac{1}{2}ab\sin\theta$. Thus, the area of the cross-section if $\displaystyle\frac{1}{2}\frac{x}{4}\frac{x}{4}\sin\frac{\pi}{3} = \frac{\sqrt{3}x^2}{64}$.

The volume is $\displaystyle\int_0^{20} \frac{\sqrt{3}x^2}{64} dx = \frac{\sqrt{3}x^3}{64 * 3}\Big|_0^{20} = \frac{125\sqrt{3}}{3} m^3$

**23.** $\displaystyle\frac{dx}{dt} = 6t; \frac{dy}{dt} = 6t^2.$

$\displaystyle L = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2} dt = 6\int_0^2 \sqrt{t^2 + t^4} dt = 6\int_0^2 t\sqrt{1 + t^2} dt = 6\frac{1}{2}\int_1^5 \sqrt{u} du =$
$\displaystyle 3\,\frac{2}{3}u^{3/2}\Big|_1^5 = 2(5\sqrt{5} - 1)$

using the substition $u = 1 + x^2$, $du = 2x\, dx$ (hence $x\, dx = \frac{1}{2}du$), and $u = 1$ when $x = 0$, $u = 5$ when $x = 2$.

**25.** $\displaystyle\frac{dy}{dx} = \frac{1}{6}\frac{3}{2}(x^2 + 4)^{1/2}(2x) = \frac{1}{2}x(x^2 + 4)^{1/2}$

$\displaystyle L = \int_0^3 \sqrt{1 + \frac{1}{4}x^2(x^2 + 4)} dx = \int_0^3 \sqrt{\frac{4 + x^4 + 4x^2}{4}} dx = \int_0^3 \sqrt{\frac{(x^2 + 2)^2}{4}} dx =$
$\displaystyle\int_0^3 \frac{x^2 + 2}{2} dx = \frac{1}{2}\int_0^3 x^2 + 2\, dx = \frac{1}{2}\left(\frac{x^3}{3} + 2x\right)\Big|_0^3 = \frac{15}{2}$

**28.** To move the elevator up 30 ft, the work of $1600 \times 30 = 48,000$ lb-ft is required.

Since we are raising a 200 ft cable only 30 ft up, 170 ft of it will be raised all the way. The work required is $170 \times 10 \times 30 = 51,000$ lb-ft.

To compute the work required to lift the remaining 30 ft of cable, we split it into pieces. To lift a segment of the cable of length $\Delta x$ up $x$ feet, the work of $\Delta x \times 10 \times x$ is required. Hence, the total work is $\int_0^{30} 10x\,dx = 5x^2|_0^{30} = 4,500$ lb-ft.

Added up, the work is $48,000 + 51,000 + 4,500 = 103,500$ lb-ft.

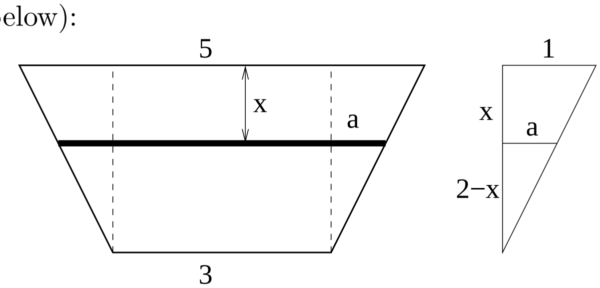
**29a.** First, determine what kind of a parabola is rotated to obtain the tank's shape. We can assume that the parabola's vertex is at the origin, i.e. that it is of the form $y = ax^2$. Since when $x = 4$, $y = 4$ (at the top), we see that $4 = a4^2$, i.e. $a = 1/4$.

At the distance of $y$ ft from the bottom, the horizontal cross-section is a circle with radius $x = 2\sqrt{y}$ (because the tank's shape is determined by the graph of $y = x^2/4$). Hence, its area is $\pi x^2 = 4\pi y$. The cross-section can be thought of as a thin layer of volume $4\pi y \Delta y$. Its weight is $62.5(4\pi y \Delta y)$. The distance to the top is $4 - y$, hence the work required to lift the layer to the top is $62.4(4\pi y \Delta y)(4 - y)$.

$W = \int_0^4 250\pi y(4-y)\,dy = 250\int_0^4 4y - y^2\,dy = 250\left(2y^2 - \frac{y^3}{3}\right)\Big|_0^4 = \frac{8000\pi}{3}$ lb-ft.

**31.** Assume that the gate is made of strips with height $\Delta x$. If the area of the strip at the depth of $x$ feet is $A(x)$, then the hydrostatic force acting on the strip is $62.5A(x)x$ lb.

$A(x)$ is the area of a rectangle with width $\Delta x$ and length $3 + 2a$ (see below):




To determine, consider the triangle on the right. The smaller and the larger triangles are similar, hence $\frac{a}{1} = \frac{2-x}{2}$. Thus $a = (2-x)/2$. Therefore, $A(x) = (3 + 2a)\Delta x = (3 + (2 - x))\Delta x = (5 - x)\Delta x$.

The hydrostatic force acting on the strip is then $62.5(5-x)x\Delta x$. Summing up the forces and passing to the limit, we have

$F = \int_0^2 62.5(5-x)x\,dx = 62.5\int_0^2 (5x - x^2)dx = 62.5\left(\frac{5x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2 = 62.5\left(10 - \frac{8}{3}\right) = \frac{1375}{3} = 458\frac{1}{3}$ lb.

**37.** (a) The function $f(x)$ is never negative. The only other condition to check is $\int_{-\infty}^{\infty} f(x)dx = 1$:

$\int_{-\infty}^{\infty} f(x)dx = \int_0^{10} \frac{\pi}{20}\sin\left(\frac{\pi x}{10}\right) dx = \int_0^{\pi} \frac{1}{2}\sin u\, du = \frac{1}{2}(-\cos u)|_0^{\pi} = \frac{1}{2}(-\cos\pi - (-\cos 0)) = \frac{1}{2}(-(-1) - (-1)) = 1$

using the substitution $u = \frac{\pi}{10}x$, $du = \frac{\pi}{10}dx$, and $u = 0$ when $x = 0$, $u = \pi$ when $x = 10$.

(b) $P(X < 4) = \int_{-\infty}^{4} f(x)\, dx = \int_0^4 \frac{\pi}{20}\sin\left(\frac{\pi x}{10}\right) dx = \int_0^{2\pi/5} \frac{1}{2}\sin u\, du = \frac{1}{2}(-\cos u)|_0^{2\pi/5} = \underline{\frac{1-\cos(2\pi/5)}{2}}$

using same substitution as in (a)

(c) The mean is $\mu = \int_0^{10} x\frac{\pi}{20}\sin\left(\frac{\pi x}{10}\right) dx = -\frac{1}{2}x\cos\left(\frac{\pi x}{10}\right)\Big|_0^1 0 + \frac{1}{2}\int_0^{10} \cos\left(\frac{\pi x}{10}\right) dx = -\frac{1}{2}x\cos\left(\frac{\pi x}{10}\right)\Big|_0^{10} + \frac{1}{2}\,\frac{10}{\pi}\sin\left(\frac{\pi x}{10}\right)\Big|_0^{10} = -\frac{1}{2}\left(10\cos\pi\right) = 5$

(using integration by parts with $x = u, du = dx, v = -\frac{1}{2}\cos\left(\frac{\pi x}{10}\right), dv = \frac{\pi}{20}\sin\left(\frac{\pi x}{10}\right) dx$)

That $\mu = 5$ is to be expected since $f(x)$ is symmetric about $x = 5$, hence this is where the mean value lies.