RESEARCH STATEMENT

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My interests lie in the areas of topology and geometry and their interaction with quantum field theory (QFT). My research focuses on string structures, which induce spin structures on the free loop space of a manifold. String structures are related to constructions in QFT and string theory, but they also arise naturally in algebraic topology. While physicists usually ignore the choice of a specific string structure, topology has hinted that there should be refined analytic and geometric invariants which depend on the particular choice.

My research attempts to better understand how things depend on the choice of a string structure. In past research I described the harmonic 3-forms on principal bundles and related this to string structures. Using this information, to a Riemannian manifold (M, g) with string structure $\mathcal{S}$ we associate a *canonical* 3-form $H_{g,\mathcal{S}} \in \Omega^3(M)$. Current research investigates a hypothesis of mine that states the following: if a manifold with string structure admits a metric such that the Ricci curvature is positive and the canonical 3-form $H_{g,\mathcal{S}} = 0$, then the cohomology class $\sigma(M, \mathcal{S}) \in tmf^{-n}(pt)$ vanishes. Other current research attempts to generalize the calculation of harmonic k-forms to $k > 3$ and relate these differential forms to fields appearing in M-Theory, a unifying string theory.

1. Background and relevance of string structures.

Let M be an n-dimensional compact oriented manifold without boundary. The existence of a string structure is purely topological, and M admits a ***string structure*** if it admits a spin structure and the Pontryagin class $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$. If a spin manifold admits a string structure, then there are $H^3(M;\mathbb{Z})$ many different isomorphism classes of string structures. These properties follow from defining $BString(n) \to BSpin(n)$ to be the homotopy fiber of the stable characteristic class $\frac{p_1}{2} : BSpin(n) \to K(\mathbb{Z}, 4)$. The following situation, reminiscent of index theory on finite-dimensional manifolds, demonstrates the importance of string structures.

There is a topological invariant $\phi_W(M)$ known as the ***Witten genus***, and while it is defined for any oriented n-manifold, the existence of a string structure implies $\phi_W(M)$ is an integral modular form of weight $n/2$ or degree n, denoted $\phi_W(M) \in MF_n$. While the modularity and integrality can be proven computationally, Witten explans in [Wit1] that the underlying reason for these properties is that $\phi_W(M)$ is the S^1-equivariant index of the Dirac operator $\not{D}_{LM}$ on the free loop space LM,

$$\phi_W(M) = \text{index}^{S^1}(\not{D}_{LM}).$$

Unfortunately, the above index theorem is only heuristic due to analytic difficulties in mathematically defining $\not{D}_{LM}$. However when $\frac{p_1}{2}(M) = 0$, the spinor bundle S_{LM} can be rigorously constructed, and the choice of a string structure on M gives a spin structure on LM [CP]. Once the spinor bundle is constructed the operator $\not{D}_{LM}$ can be written formally, though the actual space of spinors on which $\not{D}_{LM}$ acts is not defined.

While $\phi_W(M)$ is independent of the choice of string structure, there is an invariant which does depend on the particular choice. The "universal elliptic cohomology theory" ***topological modular forms (tmf)***, discovered after Witten's construction, has coefficients $tmf^*(pt)$ related to modular forms MF_*. There is a homomorphism $tmf^{-*}(pt) \to MF_*$ which is a rational isomorphism, but whose kernel contains many interesting torsion groups related to homotopy theory and number theory [Hop]. Furthermore, tmf has a string orientation lifting the Witten genus [AHS]. This implies that an n-manifold M with string structure $\mathcal{S}$ naturally produces an element $\boldsymbol{\sigma(M,\mathcal{S})} \in \boldsymbol{tmf^{-n}(pt)}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & tmf^{-n}(pt) \\ & \overset{\sigma}{\nearrow} & \downarrow \\ \Omega_n^{String}(pt) & \xrightarrow{\phi_W} & MF_n\,. \end{array}$$

The Witten genus $\phi_W(M)$ only depends on the $String$-bordism class of M in $\Omega_n^{String}(pt)$.

As noted earlier, the above situation is very similar to the important Atiyah–Singer index theorem, where the Dirac operator on a spin manifold has a Clifford-index living in KO-theory that refines the $\widehat{A}$-genus. In fact, the Dirac operators on M and LM both have interpretations using QFTs. Furthermore, there is work by Stolz–Teichner in [ST] relating KO-theory to 1-dimensional QFTs and attempting to relate tmf to 2-dimensional QFTs. Other similar attempts appear in [Seg, HK, BDR]. While it is believed there should be a geometric/analytic way of refining $\not{D}_{LM}$ to give torsion classes in tmf, this is not currently understood; the construction of tmf is completely homotopy-theoretic. My research focuses on better understanding the role of the string structure and relating geometry to refined invariants like σ.

In [Sto], Stolz conjectures that if M admits a string structure and positive Ricci curvature metric, then $\phi_W(M) = 0$. This conjecture is analogous to Lichnerowicz' theorem for spin manifolds, which states that any manifold admitting a spin structure and positive scalar curvature metric must satisfy $\widehat{A}(M) = 0$. For spin manifolds, an even stronger theorem holds: if M is spin and admits a positive scalar curvature metric, then the KO-theory index is also 0 for all spin structures. Continuing the analogy between KO and tmf, a natural question is whether the invariant $\sigma(M,\mathcal{S}) \in tmf^{-n}(pt)$ vanishes for all manifolds with string structure $(M,\mathcal{S})$ admitting positive Ricci curvature metrics. In section 3, we will see why this cannot be true. We are forced to also consider the string structure if we wish to make any reasonable hypothesis concerning the vanishing of $\sigma(M,\mathcal{S})$. We will now construct a canonical 3-form $H_{g,\mathcal{S}}$ associated to any Riemannian manifold (M,g) with string structure $\mathcal{S}$. We use this 3-form $H_{g,\mathcal{S}}$ when hypothesizing a relationship between positive Ricci curvature and the σ-invariant.

2. Past research: Harmonic forms and string structures

Let M be a spin manifold, and consider the principal $Spin(n)$-bundle $Spin(TM) \xrightarrow{\pi} M$. The bundle $Spin(TM)$ is a manifold locally diffeomorphic to $U \times Spin(n)$ for small $U \subset M$. In [Red1] we show that, up to homotopy, the choice of a string structure is equivalent to the choice of a ***string class***, a cohomology class $\mathcal{S} \in H^3(Spin(TM);\mathbb{Z})$ that restricts fiberwise to the standard class in $H^3(Spin(n);\mathbb{Z})$. While there are different explicit models for string structures (see [Sto, ST, BSCS, Hen]), the string class $\mathcal{S}$ naturally appears in any such model.

Passing from $H^3(Spin(TM);\mathbb{Z}) \to H^3(Spin(TM);\mathbb{R})$, we lose torsion information but gain the ability to represent string classes by differential forms using Hodge theory. Choosing a Riemannian metric g on M defines the Levi–Civita connection and decomposes the tangent bundle of $Spin(TM)$ into "horizontal" and "vertical" directions. This decomposition allows us to construct a local product metric on $Spin(TM)$ using g and a bi-invariant metric on $Spin(n)$. We actually need to take an adiabatic limit, which means we globally rescale the metric g so that it is very large with respect to the metric on $Spin(n)$. In [Red2], we calculate the harmonic 3-forms in the adiabatic limit on principal bundles with metrics of the type just described. The calculation relies on a Hodge-theoretic version of the Leray–Serre spectral sequence, developed in [MM, Dai, For]. The adiabatic-harmonic form $[\mathcal{S}]$ representing $\mathcal{S} \in H^3(Spin(TM);\mathbb{R})$ decomposes as

$$[\mathcal{S}] = CS_3(g) - \pi^* H_{g,\mathcal{S}} \in \Omega^3(Spin(TM)),$$

where $CS_3(g)$ is the Chern–Simons 3-form associated to g and $H_{g,\mathcal{S}} \in \Omega^3(M)$. Therefore, once a string structure and metric are chosen, we obtain a **canonical 3-form $\boldsymbol{H_{g,\mathcal{S}} \in \Omega^3(M)}$**. The form $H_{g,\mathcal{S}}$ depends on both the metric and string structure, it encodes the Chern–Simons numbers on 3-cycles, and $dH_{g,\mathcal{S}}$ equals the Chern–Weil 4-form $\frac{p_1}{2}(M,g)$ [Red3].

These forms, both $H_{g,\mathcal{S}}$ and $CS_3(g) - H_{g,\mathcal{S}}$, are used in the construction of the spinor bundle on LM, and they also arise when constructing a QFT interpretation of the Witten genus, as noted in [Wit2, AS]. The important property gained in our construction is that $H_{g,\mathcal{S}}$ integrates to give elements in $\mathbb{R}$ as opposed to $\mathbb{R}/\mathbb{Z}$. We believe keeping track of the form $H_{g,\mathcal{S}}$ and its $\mathbb{R}$-valued periods is important in understanding subtle invariants like σ in tmf.

3. Current and future research

3.1. Positive Ricci curvature and tmf.

There exist a number of manifolds M admitting positive Ricci curvature metrics and string structures $\mathcal{S}$ such that $\sigma(M,\mathcal{S}) \neq 0$ even though $\phi_W(M) = 0$. The easiest example is $S^3 \cong SU(2)$, which even admits a metric of positive sectional curvature. Using different string structures, σ maps $(SU(2),\mathcal{S})$ surjectively onto $tmf^{-3}(pt) \cong \mathbb{Z}/24$. It is therefore impossible to hope that positive Ricci curvature always means σ is zero. However, by simultaneously considering the geometry and the string structure, we give the following hypothesis.

Hypothesis. *Let M be a spin n-manifold with choice of string structure $\mathcal{S}$. If there exists a metric g such that both $Ric(g) > 0$ and $H_{g,\mathcal{S}} = 0$, then $\sigma(M,\mathcal{S}) = 0 \in tmf^{-n}(pt)$.*

Heuristically, this hypothesizes that if no additional $H_{g,\mathcal{S}}$-term is needed to construct $\not{D}_{LM}$, then positive Ricci curvature implies the vanishing of any refined invariants. Also, the condition that both $Ric(g) > 0$ and $H_{g,\mathcal{S}} = 0$ is equivalent to requiring that a canonical metric connection with torsion determined by $H_{g,\mathcal{S}}$ has positive Ricci curvature in a scaling limit. The hypothesis holds for the previously described situation of $SU(2)$ with its standard metric and various string structures. In fact, the hypothesis holds for specific families of metrics on $SU(2)$ such as the 1-parameter family of Berger metrics given by shrinking the fibers in the Hopf fibration. The Berger metrics also demonstrate that the hypothesis would not hold if $Ric(g) > 0$ were replaced by $Ric(g) \geq 0$. The statements in this paragraph are proved in [Red3].

Investigating the hypothesis is a long-term project that leads to many interesting questions. For example, it is unclear when the conditions of the hypothesis can be satisfied. Requiring $H_{g,\mathcal{S}} = 0$ implies that the Pontryagin form $\frac{p_1}{2}(M,g)$ is identically zero. I am therefore interested in the following questions. When can one find a connection on $Spin(TM)$ so that the $\frac{p_1}{2}$-form is 0? Does there exist a Riemannian metric such that the $\frac{p_1}{2}$-form of the Levi–Civita connection is 0? If the form $\frac{p_1}{2}(M,g) = 0$, how strong is the further condition $H_{g,\mathcal{S}} = 0$? Any partial results or obstructions would be naturally interesting and applicable to other areas.

Also, in dimension 3 there is an interpretation of σ using the Adams e-invariant, and this can be calculated geometrically using the form $H_{g,\mathcal{S}}$ and a Riemannian 4-manifold that M bounds [Red3]. I have some ideas on how to test the hypothesis on arbitrary Riemannian 3-manifolds and am hopeful this will lead to conclusive results in dimension 3.

3.2. String/Fivebrane Duality and M-Theory.

Another project I am currently working on is understanding how harmonic k-forms on principal bundles decompose for values of $k \neq 3$ and to what extent the wedge product preserves this decomposition. For $k = 1$ this was calculated and related to SU-structures on a principal $U(n)$-bundle in [Red1]. For $k = 7$, the calculation of harmonic forms is still work in progress, but there is a relationship with the "fivebrane structure" defined in [SSS], a higher analog of the string structure whose obstruction is given by the degree 8 Pontryagin class $\frac{1}{6}p_2(M) = 0 \in H^8(M;\mathbb{Z})$.

In joint work with Hisham Sati, a mathematical physicist at Yale, we are currently relating string structures and their differential form representatives to the M-theory C-field. Very briefly, the C-field is a higher analog of a bundle with connection and is locally a 3-form with a globally well-defined fieldstrength given by degree 4 characteristic classes and forms [DMF]. The dual C-field, on an 11-dimensional manifold, is locally a 7-form with well-defined field strength given by degree 8 characteristic classes and forms.

We hope to relate string structures with the C-field, relate fivebrane structures with the dual C-field, and express the C-field duality in terms of a string/fivebrane duality.

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