MATH 552 HOMEWORK 3. DUE 11/7

(1) (a) Show that the symmetric power representations $S^k\mathbb{C}^2$ are isomorphic, as $sl(2,\mathbb{C})$ representations, to the irreducible representations V_k of highest weight k.
   (b) Write $(C^2)^{\otimes 3}$ as a sum of irreducible $sl(2,\mathbb{C})$ representations.
   (c) Write $S^2\mathbb{C}^2 \otimes S^3\mathbb{C}^2$ as a sum of irreducible $sl(2,\mathbb{C})$ representations. Do the same for $S^2\mathbb{C}^2 \otimes S^5\mathbb{C}^2$. (Do you notice a pattern?)

(2) Decompose $(\mathbb{C}^3)^{\otimes 2} \otimes (\mathbb{C}^{3*})$ as a sum of irreducible $sl(3,\mathbb{C})$ representations. You may use the fact that the irreducible representation $\Gamma_{2,1}$ of highest weight $(2,1)$ has dimension 15. You can print off triangular graph paper at http://incompetech.com/graphpaper/triangle/

(3) In last homework (problem 4.5), you showed that if V is an irreducible representation of $\mathfrak{g}$, then the space of invariant symmetric bilinear forms on V is either 0 or 1-dimensional, depending on whether $V \cong V^*$ as $\mathfrak{g}$-representations.
   (a) (5.6) If $\mathfrak{g}$ is a simple Lie algebra, show that the symmetric invariant bilinear form is unique up to a factor and that $\mathfrak{g} \cong \mathfrak{g}^*$ as representations of $\mathfrak{g}$.
   (b) Show that any irreducible $sl(2,\mathbb{C})$ representation admits a non-zero invariant bilinear form.
   (c) Show that this is not true for $sl(3,\mathbb{C})$ representations.
   (d) Characterize, using the weights, when an irreducible $sl(3,\mathbb{C})$ representation is isomorphic to its dual representation (i.e. self-dual).

(4) (Optional) 5.3 Show that for $\mathfrak{g} = sl(n,\mathbb{C})$, the Killing form is given by

$$K(x,y) = 2n\mathrm{tr}(xy)$$

where tr is the usual trace of the matrix representation.