MATH 552 HOMEWORK 2 REVISED. DUE 10/17

1. Let N be the subgroup of real upper-triangular 3×3 matrices with 1's on the diagonals; i.e. an element of N is of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{R}.$$

Let Z be the discrete normal subgroup (isomorphic to $\mathbb{Z}$) of matrices of the form

$$\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n \in \mathbb{Z}.$$

The group $H = N/Z$ is called the Heisenberg group (sometimes N is called the Heisenberg group).

(1) Show that the Hilbert space $L^2(\mathbb{R})$ (of $\mathbb{C}$-valued functions) is naturally a representation of H by

$$\left(\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} f \right)(x) := e^{-2\pi i c} e^{2\pi i b x} f(x - a)$$

(2) **Changed** Show that S^1 is naturally a compact abelian subgroup of H, and there is a short exact sequence

$$1 \to S^1 \to H \to \mathbb{R}^2 \to 1.$$

(3) Show that H has no faithful finite-dimensional representation, and therefore H is not isomorphic to a matrix group.

Hint: Use the S^1 subgroup and Lemma 4.21 in Kirillov to decompose a representation V of H. For any element in S^1, what will be the determinant of its action on V? It will help to show that any element of the central S^1 subgroup is in the commutator subgroup $[H, H]$ (where $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ is the group commutator).

Kirillov: 3.15, 4.5, 4.8