Math 531 Midterm Review Solutions

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1.$$

For which points $p = (0, 0), p = (\frac{1}{3}, \frac{1}{3}), p = (\frac{-1}{3}, \frac{-1}{3})$, is $f^{-1}(f(p))$ an embedded submanifold in $\mathbb{R}^2$?

The answer here is quite subtle. The points $p = (1/3, 1/3)$ and $p = (-1/3, -1/3)$ give submanifolds, while $p = (0, 0)$ does not. However, $p = (-1/3, -1/3)$ does not follow from the Implicit Function Theorem.

Let us first do a few calculations:

$$\begin{aligned} f(x, y) &= x^3 + xy + y^3 + 1 \\ f(0, 0) &= 1 \\ f(1/3, 1/3) &= 32/27 \\ f(-1/3, -1/3) &= 28/27 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 0 & 0 \end{bmatrix} = Df &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 3x^2 + y & x + 3y^2 \end{bmatrix} \end{aligned}$$

The two points where Df has rank 0 (where Df does not have full rank) occur at $(0, 0)$ and $(-1/3, -1/3)$. Therefore, since f has full rank at all points $M = f^{-1}(32/37)$, we know that M is a codimension 1 submanifold of $\mathbb{R}^2$.

However, we do not yet know whether the other two sets are submanifolds or not. The above process merely guarantees a submanifold structure in certain situations; it does eliminate the possibility. In general, there is no standard method for showing that something is not a manifold, though usually you assume you have a submanifold and try to obtain some sort of contradiction. We investigate the two unknown situations from above more carefully.

First, let us consider, $f^{-1}f(-1/3, -1/3)$. Some sneaky algebra (it helps to use a computer) shows us that

$$\begin{aligned} x^3 + xy + y^3 + 1 &= 28/27 \\ x^3 + xy + y^3 - 1/27 &= 0 \\ \frac{1}{27}(3x + 3y - 1)(9x^2 + 3x - 9xy + 3y + 9y^2 + 1) &= 0 \end{aligned}$$

We know that $3x + 3y - 1 = 0$ is a submanifold of $\mathbb{R}^2$. Further investigation shows that the second factor

$$9x^2 + 3x - 9xy + 3y + 9y^2 + 1 \geq 0.$$

The above follows from a bit of Calc 3. We can show that this quadratic function has a critical point at $(-1/3, -1/3)$, $f(-1/3, 1/3) = 0$, and the Hessian is positive definite (f is "concave up" in every direction). Therefore,

$$\begin{aligned} f^{-1}f(-1/3,-1/3) &= \{(x,y)|3x+3y-1=0\} \cup (-1/3,-1/3) \\ &= \{(x,y)|3x+3y-1=0.\} \end{aligned}$$

So, this particular subset is a submanifold, though you cannot simply use the Implicit Function Theorem.

On the other hand, $f^{-1}f(0,0)$ is not a manifold because of the point $(0,0)$. Seeing this is a little trickier, and I won't expect you to do anything like this on the exam. I'll try to write up a clean solution to show this is not, but this is not the important part of the problem. It is, though, good to see that these things can happen.

2. Let M be a compact manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a smooth map. Then f must have at least one critical point.

Proof. Suppose there are no critical points. Then, by the Inverse Function Theorem, for every $p \in M$, there exists an open neighborhood U of p such that $f_{|U}$ is a diffeomorphism. This implies that f is an open map (it takes open sets to open sets), since these open neighborhoods form a basis for the open sets in M. Therefore, $f(M) \subset \mathbb{R}^n$ is open subset. However, M is compact, and therefore the image $f(M) \subset \mathbb{R}^n$ is also compact. We have now reached a contradiction, as there are no open compact subsets of $\mathbb{R}^n$ (other than the empty set). The map f must have at least one critical point. □

3. Let S^2 be the 2-sphere. Let $U_N = S^2 - \{N\}, U_S = S^2 - \{S\}$ be the open sets obtained by removing the "North Pole" and the "South Pole," respectively. On both U_N and U_S there exist standard stereographic projections to $\mathbb{R}^2$ (dealt with in a previous homework). These coordinate charts give a trivialization of the tangent bundle over each open set. QUESTION: Compute the transition functions for the tangent bundle on the overlap. i.e. if ϕ_N, ϕ_S are the local trivializations of TS^2 induced by the two stereographic projections, calculate $\phi_S \circ \phi_N^{-1}$.

First, let us set up the following notation:

$$\begin{aligned} S^2 &= \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2=1\} \\ U &= U_N, V = U_S \\ (u^1,u^2) &= (\frac{x}{1-z}, \frac{y}{1-z}) = u(x,y,z) \\ (v^1,v^2) &= (\frac{x}{1+z}, \frac{y}{1+z}) = v(x,y,z) \end{aligned}$$

Here, u and v are the standard coordinates induced from the stereographic projection, and on overlap $U \cap V$, we have functions $u \circ v^{-1}$ and $v \circ u^{-1}$ given by the relations

$$v^i = \frac{u^i}{(u^1)^2 + (u^2)^2}$$
$$u^i = \frac{v^i}{(v^1)^2 + (v^2)^2}$$

The two trivializations u, v both give rise to a trivialization of the tangent bundle over U and V, respectively. Explicitly, this is an equivalence of bundles ϕ_U

$$U \times \mathbb{R}^2 \overset{\phi_U}{\longleftarrow} TU$$
$$(u, (a^1, a^2)) \longmapsto a^1 \frac{\partial}{\partial u^1} + a^2 \frac{\partial}{\partial u^2}$$

along with the similarly defined equivalence ϕ_V. The transition function $\phi_V \circ \phi_U^{-1}$ is then a bundle equivalence

$$\phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{R}^2 \to (U \cap V) \times \mathbb{R}^2.$$

Since $\phi_V \circ \phi_U^{-1}$ is a bundle map, we can consider this as a smooth map

$$\phi_V \circ \phi_U^{-1} : U \cap V \to Gl(2, \mathbb{R}).$$

Remembering that the same vector X can be expressed as

$$X = a^i \frac{\partial}{\partial u^i} = b^i \frac{\partial}{\partial v^i},$$

we obtain the standard relationship

$$b^j = \frac{\partial v^j}{\partial u^i} a^i.$$

(Remark: this can be seen directly from the chain rule, which states

$$\frac{\partial}{\partial v^j} = \frac{\partial u^i}{\partial v^j} \frac{\partial}{\partial u^i}. \quad)$$

Now, we see that the transition function $\phi_V \circ \phi_U^{-1}$, which maps

$$(a^1, a^2) \overset{\phi_V \circ \phi_U^{-1}}{\longmapsto} (b^1, b^2),$$

is given by the linear operators (matrices)

$$\phi_V \circ \phi_U^{-1}(p) = \left[\frac{\partial v^i}{\partial u^j} \right]_{|p} = D(v \circ u^{-1})_p.$$

A simple calculation now shows us that

$$
\begin{aligned}
\frac{\partial v^i}{\partial u^j} &= \frac{(\sum_i (u^i)^2)\delta^i_j - 2u^i u^j}{(\sum_i (u^i)^2)^2} \\
&= \left(\frac{1}{\sum_i (u^i)^2}\right)^2 \begin{bmatrix} -(u^1)^2 + (u^2)^2 & -2u^1u^2 \\ -2u^1u^2 & (u^1)^2 - (u^2)^2 \end{bmatrix} \\
&= \left(\frac{1}{1+z}\right)^2 \begin{bmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix}
\end{aligned}
$$

So, we have the transition function $g_{VU} = \phi_V \circ \phi_U^{-1} : (U \cap V) \to Gl(2, \mathbb{R})$ is given by

$$
g_{VU} = \left(\frac{1}{\sum_i (u^i)^2}\right)^2 \begin{bmatrix} -(u^1)^2 + (u^2)^2 & -2u^1u^2 \\ -2u^1u^2 & (u^1)^2 - (u^2)^2 \end{bmatrix} = \left(\frac{1}{1+z}\right)^2 \begin{bmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix},
$$

in u coordinates and (x, y, z) coordinates, respectively (one may be easier to use than the other). We can also compute $g_{UV} = g_{VU}^{-1} = \phi_U \circ \phi_V^{-1}$. This can be done by performing the above calculation for $\dfrac{\partial u^i}{\partial v^j}$, or by taking the inverse of the matrix g_{VU} above. Either way, we end up with

$$
g_{UV} = \left(\frac{1}{\sum_i (v^i)^2}\right)^2 \begin{bmatrix} -(v^1)^2 + (v^2)^2 & -2v^1v^2 \\ -2v^1v^2 & (v^1)^2 - (v^2)^2 \end{bmatrix} = \left(\frac{1}{1-z}\right)^2 \begin{bmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix}
$$

4. The Klein bottle is not orientable.

Proof. The Klein bottle contains the open Mobius band as a submanifold. We know that if a manifold M contains a non-orientable submanifold, then the manifold M is not orientable.

Alternatively, we can think of the Klein bottle M as a quotient of the cylinder. In other words, $(\mathbb{R} \times S^1)/\mathbb{Z}$, where $\mathbb{Z}$ acts by the group action

$$
(x, \theta) \sim (x + m, (-1)^m \theta), \quad m \in \mathbb{Z}.
$$

The quotient map $\pi : \mathbb{R} \times S^1 \to M$ induces a map on the tangent bundles

$$
T(\mathbb{R} \times S^1) \xrightarrow{\pi_*} TM,
$$

and for any $m \in \mathbb{Z}$, we have the following commutative diagram

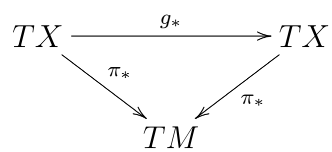
$$
\begin{array}{ccc}
T(\mathbb{R} \times S^1) & \xrightarrow{\quad g_* \quad} & T(\mathbb{R} \times S^1) \\
 & {\scriptstyle \pi_*} \searrow \quad \swarrow {\scriptstyle \pi_*} & \\
 & TM &
\end{array}
$$

Now, we notice that the map

$$(x, \theta) \mapsto (x + 1, -\theta),$$

which is the action of $1 \in \mathbb{Z}$, reverses the orientation of $\mathbb{R} \times S^1$. (This is an easy calculation.) Therefore, by the lemma below, M is not orientable. $\square$

Lemma 1. *Let X is a connected orientable manifold with map $g : X \to X$, where g is orientation-reversing. If there exists a local diffeomorphism $\pi : X \to M$ such that the following diagram commutes*



then M is not orientable.

In particular, if a discrete group G acts on X and there is some $g \in G$ that reverses the orientation, then X/G is non-orientable.

Proof. First, notice that if M is orientable, and X and M are both oriented, then π_* must either preserve orientation at every point, or it must reverse orientation at every point.

Now, let $x \in X$, $p = \pi(x)$, and choose an orientation on X. Notice that

$$\pi_{*|x} : T_x X \to T_p M$$

either preserves or reverses orientation. However,

$$(\pi_* \circ g_*)_{|x} : T_x X \to T_p M$$

does the opposite of π_{*x}. By the commutativity of the diagram, $\pi_{*|x} = (\pi_* \circ g_*)_{|x}$, and thus M cannot admit an orientation. $\square$

5. On $\mathbb{R}^3$ with standard coordinates, consider the vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$
$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Then,

$$[X, Y] = Z, [Y, Z] = X, [Z, X] = Y.$$

This is a straight-forward calculation using the definition of the Lie bracket.

$$\begin{aligned}[X,Y] &= \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right) - \left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \\ &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = Z\end{aligned}$$

Notice that, since we know the answer will be vector field, we can ignore terms with higher-order derivatives. Of course, if you just write everything out, you will see that these cancel out. Similar calculations show $[Z,X] = Y, [Y,Z] = X$.

One might notice that this structure looks a bit like the cross-product structure on $\mathbb{R}^3$ (replacing X, Y, Z with i, j, k). In fact, this can be made explicit. The 3-dimensional vector space spanned by the vector fields $\{X, Y, Z\}$, together with the Lie bracket, form a Lie algebra. Likewise, $\mathbb{R}^3$ with the cross-product structure also forms a Lie algebra. These two Lie algebras are isomorphic under the map

$$aX + bY + cZ \longmapsto a\widehat{i} + b\widehat{j} + c\widehat{k}.$$

The above map is linear and invertible, and the two brackets commute with each other (check this).

Furthermore, it is not too difficult to show that the flow generated by $aX + bY + cZ$ is actually a 1-parameter subgroup of $SO(3,\mathbb{R})$. This can be seen through the work done in homework problem Ch. 5, number 6, as well as a bit of what we talked about in class concerning matrix groups.