Math 531 Midterm Review

The midterm will be a timed, take-home exam (taken on the honor system). You will have 3 hours to complete the exam. You may use your notes and Spivak, but you may not obtain outside help from anyone. The problems will be a mix of abstract theorems and concrete calculations. There will be multiple problems, and you will be able to choose which ones you do (e.g. work 2 out of 3 problems). Here are a few good problems to practice with:

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1.$$

For which points $p = (0, 0), p = (\frac{1}{3}, \frac{1}{3}), p = (\frac{-1}{3}, \frac{-1}{3})$, is $f^{-1}(f(p))$ an embedded submanifold in $\mathbb{R}^2$?

2. Let M be a compact manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be a smooth map. Does f have to have a critical point? In other words, must there exist a point $p \in M$ such that f_{*p} is not injective?

3. Let S^2 be the 2-sphere. Let $U_N = S^2 - \{N\}, U_S = S^2 - \{S\}$ be the open sets obtained by removing the "North Pole" and the "South Pole," respectively. On both U_N and U_S there exist standard stereographic projections to $\mathbb{R}^2$ (dealt with in a previous homework). These coordinate charts give a trivialization of the tangent bundle over each open set. QUESTION: Compute the transition functions for the tangent bundle on the overlap. i.e. if ϕ_N, ϕ_S are the local trivializations of TS^2 induced by the two stereographic projections ($\phi_N : TU_N \to U_N \times \mathbb{R}^2$, $\phi_S : TU_S \to U_S \times \mathbb{R}^2$), calculate $\phi_S \circ \phi_N^{-1}$.

4. Is the Klein bottle orientable? Justify your answer.

5. On $\mathbb{R}^3$ with standard coordinates, consider the vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$
$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Calculate the Lie bracket on the vector fields X, Y, Z. What sort of structure does this look like?