Math 531 Midterm Solutions

1. In multivariable calculus, given a function $f : \mathbb{R}^n \to \mathbb{R}$, you define the gradient vector ∇f by

$$\nabla f = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

In other words, it is vector field with component functions $\dfrac{\partial f}{\partial x^i}$.

Consider the more general situation where M is a smooth manifold, and $f : M \to \mathbb{R}$ a smooth function.

a. Show that the functions $\dfrac{\partial f}{\partial x^i}$ (where x is a local coordinate system) do not form the component functions of a vector field. Show, however, that they do naturally form the component functions of a 1-form.

b. Suppose we have a tensor $g \in C^\infty(M, TM^* \otimes TM^*)$, and further suppose that at each point $x \in M$, the tensor g_x is symmetric and non-degenerate. In other words,

$$g_x(v_1, v_2) = g_x(v_2, v_1),$$

and the rewritten

$$g_x : T_xM^{**} \to T_xM^*$$

is invertible. Then, **show** that g can be used to change the components $\frac{\partial f}{\partial x^i}$ into components of a vector field.

Solution:

Notice that we can define $df \in C^\infty(M, TM^*)$ in a coordinate-free way. That is, $df(X) = X(f)$, where X is a vector field. Then, a simple calculation shows that

$$df = \frac{\partial f}{\partial x^i} dx^i$$

in local coordinates. This implies $\frac{\partial f}{\partial x^i}$ are the components of a (non-trivial) 1-form, and hence not a vector field.

More explicitly, suppose a^i, b^i are the component functions of a vector field in x, y coordinates, respectively. Then these are related by the equation

$$b^j = a^i \frac{\partial y^j}{\partial x^i}.$$

However, we see that

$$\frac{\partial f}{\partial y^j} = \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j}.$$

This does not satisfy the transformation property for components of vector fields, but instead satisfies the transformation property for differential forms.

In fact, we can see that that

$$\frac{\partial f}{\partial x^i} dx^i = \left(\frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i} \right) \left(\frac{\partial x^i}{\partial y^j} dy^j \right) = \frac{\partial f}{\partial y^j} dy^j,$$

whereas

$$\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} = \frac{\partial f}{\partial y^j}\frac{\partial}{\partial y^j}\left(\frac{\partial x^i}{\partial y^j}\right)^2 \neq \frac{\partial f}{\partial y^j}\frac{\partial}{\partial y^j}.$$

Therefore, $\frac{\partial f}{\partial x^i}dx^i$ is a well-defined 1-form on M, but not a well-defined vector field.

Now, suppose we have the tensor g. Such a tensor g is a non-degenerate inner product on each tangent space and is called a metric (if g is positive-definite, then it is a Riemannian metric). (Remark: we don't need the symmetric property here, but it makes indices a little easier to deal with.)

If $g \in C^\infty(M, TM^* \otimes TM^*)$, then we can equivalently consider

$$g \in C^\infty(M, \mathrm{Hom}(TM, TM^*))$$

(using the generic natural equivalences $\mathrm{Hom}(V, W) \cong V^* \otimes W$ and $V^{**} \cong V$). The non-degeneracy of g implies that we have

$$g^{-1} \in C^\infty(M, \mathrm{Hom}(TM^*, TM)) \cong C^\infty(M, TM \otimes TM).$$

Therefore, we can construct $\nabla f = g^{-1}(df)$ by the following:

$$\begin{array}{ccccc} C^\infty(M) & \xrightarrow{d} & C^\infty(M, TM^*) & \xrightarrow{g^{-1}} & C^\infty(M, TM) \\ f & \longmapsto & df & \longmapsto & g^{-1}(df) = \nabla f \end{array}$$

For fun, let's write this in local coordinates. We know that

$$g = g_{ij} dx^i \otimes dx^j.$$

Since g is non-degenerate, the matrix g_{ij} is invertible. Hence, we also have the tensor

$$g^{-1} = g^{ji}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

We now see that we can use g^{-1} to transform the above df into a vector field. We define

$$\begin{aligned} \nabla f = g^{-1}(df) &= g^{kj}\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}\left(\frac{\partial f}{\partial x^i}dx^i\right) \\ &= g^{kj}\frac{\partial}{\partial x^j}\delta^i_k\frac{\partial f}{\partial x^i} \\ &= g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} \end{aligned}$$

It is simple to check that ∇f is now a well-defined vector field. Suppose that

$$g^{-1} = g^{\beta\alpha}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}.$$

Then, we see that

$$\begin{aligned}
g^{\alpha\beta}\frac{\partial f}{\partial y^\alpha}\frac{\partial}{\partial y^\beta} &= \left(g^{ij}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\right)\left(\frac{\partial f}{\partial x^k}\frac{\partial x^k}{\partial y^\alpha}\right)\left(\frac{\partial x^l}{\partial y^\beta}\frac{\partial}{\partial x^l}\right)\\
&= g^{ij}\delta_i^k\delta_j^l\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^l}\\
&= g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}
\end{aligned}$$

Therefore, we see that ∇f is a well-defined vector field.

The above demonstrates the general principle that if some quantity is written in local coordinates, each index should appear "up" and "down" an equal number of times.

In summary, we see that we can use the tensor g to transform the functions $\frac{\partial f}{\partial x^i}$ into components of a vector field. In the situation where $M = \mathbb{R}^n$ with standard coordinate x, and $g_{ij} = \delta_{ij}$ is the usual Euclidean metric, we see that

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \delta^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}.$$

2. Consider the polar coordinates on (subsets of) $\mathbb{R}^2$ defined by

$$\begin{aligned}
x &= r\cos\theta\\
y &= r\sin\theta
\end{aligned}$$

Compute $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in polar coordinates .

Solution:
We know that

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}
\end{aligned}$$

Letting u and p denote the Euclidean and polar coordinate systems, respectively, we know that

$$D(p\circ u^{-1}) = D((u\circ p^{-1})^{-1}) = [D(u\circ p^{-1})]^{-1}].$$

As easy calculation shows us that

$$D(u\circ p^{-1}) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}.$$

Therefore,

$$\begin{aligned}
D(p\circ u^{-1}) &= \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}\end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}^{-1}\\
&= \begin{bmatrix} \cos\theta & \sin\theta\\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r}\end{bmatrix}
\end{aligned}$$

Therefore, we see that

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}$$

3. In an ODE class, you often encounter the following spring equation:

$$\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt},$$

where $x(t)$ is the displacement of a mass at time t, and k, c are constants related to the strength of the spring force and resistance force, respectively.

a. Let $y = \dfrac{dx}{dt}$, and rewrite the above equation in terms of a vector field on $M = \mathbb{R}^2$.

b. Show that flow induced by the vector field in (a) is a 1-parameter subgroup of $Gl(2, \mathbb{R})$.

Solution:
First, let $y = \frac{dx}{dt}$. The 2nd-order differential equation now becomes a system of first-order equations

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -kx - cy \end{cases}$$

Remembering that we are solving for functions $(x(t), y(t))$, we think about solutions to the above equation as paths $\gamma = (\gamma^x, \gamma^y) : (-\epsilon, \epsilon) \to \mathbb{R}^2$. The equation is then equivalent to

$$\begin{cases} \frac{d\gamma^x}{dt}\frac{\partial}{\partial x} = y\frac{\partial}{\partial x} \\ \frac{d\gamma^y}{dt}\frac{\partial}{\partial y} = (-kx - cy)\frac{\partial}{\partial y} \end{cases}$$

In other words, we want

$$\gamma_*(\frac{\partial}{\partial t}) = \frac{d\gamma}{dt} = \frac{d\gamma^x}{dt}\frac{\partial}{\partial x} + \frac{d\gamma^y}{dt}\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} + (-kx - cy)\frac{\partial}{\partial y} = X_{\gamma(t)}.$$

Solutions to the system of ODEs are equivalent to integral curves to the vector field

$$X = y\frac{\partial}{\partial x} + (-kx - cy)\frac{\partial}{\partial y}.$$

The above equation is system of linear equations with constant coefficients. In homework problem number...., we saw that if this is rewritten as

$$\frac{d\gamma}{dt} = A\gamma,$$

where A has constant coefficients, then the integral curve to X with initial point $\gamma(0) = \gamma_0$ is given by

$$\gamma(t) = e^{At}\gamma_0.$$

Therefore, we have that (in standard components)

$$\gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Therefore, the integral curves, with $\gamma(0) = (x_0, y_0)$, will be of the form

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} t\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is then given by

$$\phi_t = e^{At} \in Gl(2, \mathbb{R}).$$

Furthermore, ϕ is a homomorphism $\mathbb{R} \to Gl(2, \mathbb{R})$ (where $\mathbb{R}$ is the additive group), showing that the flow is a 1-parameter subgroup of $Gl(2, \mathbb{R})$. Equivalently, we can explicitly see that

$$\phi_0 = e^{0A} = I, \quad \phi_t \circ \phi_s = e^{A(s+t)}.$$

Solve 1 of the following 2 problems:
4. Implicit differentiation is a useful tool in calculus. Essentially, if we have

$$f(x, y) = c,$$

then we solve for $\dfrac{dy}{dx}$ by implicitly differentiating and obtaining

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Prove that this is mathematically rigorous.

Solution: Suppose we wish to calculate $\frac{dy}{dx}$ at a point (x_0, y_0), and assume that $\frac{\partial f}{\partial y} \neq 0$ at (x_0, y_0). Then, by the implicit function theorem, $f(x, y) = c$ implicitly defines y as a function of x. In other words, in a neighborhood of (x_0, y_0), there exists $g(x)$ (=$y(x)$)such that

$$f(x, g(x)) = c.$$

Hence, we calculate

$$\begin{aligned} \frac{d}{dx}|_{x_0} \left(f(x, g(x))\right) &= \frac{d}{dx}|_{x_0} \left(c\right) \\ D_1 \left(f(x_0, g(x_0))\right) \frac{dx}{dx}|_{x_0} + D_2 \left(f(x_0, g(x_0))\right) \frac{dg}{dx}|_{x_0} &= 0 \end{aligned}$$

Then, using the shorthand of $y(x) = g(x)$, we obtain the standard expression

$$\begin{aligned} \frac{\partial f}{\partial x}|_{(x_0,y_0)} + \frac{\partial f}{\partial y}|_{(x_0,y_0)} \frac{dy}{dx}|_{(x_0,y_0)} &= 0 \\ -\frac{\partial f/\partial x}{\partial f/\partial y}|_{(x_0,g(x_0)} = \frac{dy}{dx}|_{(x_0,y_0)} &= \frac{dg}{dx}|_{x_0} \end{aligned}$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Notice that since wish to divide by $\frac{\partial f}{\partial y}$, we would have wanted to require $\frac{\partial f}{\partial y}$ to be non-zero.

5. Suppose M and N are connected, oriented manifolds and $f : M \to N$ is a local diffeomorphism. Show that if f_{*p} preserves orientation at some $p \in M$, then f_* preserves orientation at all points $p \in M$.

Solution: First, f is a local diffeomorphism implies that f_* is an isomorphism at every point. An isomorphism of vector spaces gives a bijection between the set of orientations on each vector space. Hence, it makes sense to talk about f_* preserving or switching orientation at each point.

The space of orientations at each point $p \in M$ is a discrete space. In fact, the space of orientations naturally forms a double cover, which we shall call $Or(M)$. That is, there is a topological space $Or(M)$ with a free $\mathbb{Z}/2$ action such that $Or(M)/\mathbb{Z}/2 = M$. If M, N are oriented, then the covers

$Or(M)$ and $Or(N)$ are disconnected. The choice of orientations M and N give us a choice of connected components in $Or(M), Or(N)$. We therefore have a continuous map

$$f_* : Or(M) \to Or(N).$$

Because of continuity, f_* cannot map a connected component to two different connected components. Therefore, f_* will preserve orientation at all points.

Alternatively, we can let $x_0 \in M$ be an arbitrary point. Assume $f_* : T_{x_0}M \to T_{f(x_0)}N$ preserves orientation. Then, let $x \in M$ be any other point. Since M is connected (and connectedness is equivalent to path connectedness for manifolds), we can choose some path

$$\begin{aligned} \gamma : [0,1] &\to M \\ 0 &\mapsto x_0 \\ 1 &\mapsto x \end{aligned}$$

connecting x_0 and x. The tangent spaces of M and N are continuously oriented, and f is a smooth map. This implies that if f_{*p} preserves orientation, then f_* preserves orientation in some neighborhood of p. We then take open cover containing the image of γ and see that f_x must preserve orientation.