Math 531 Midterm

Please abide by the following rules:

- Time Limit: 3 hours
- You are allowed 1 untimed break
- This test is open notes and open Spivak. You may not consult any additional sources (or people)
- You are to answer 2 of the first 3 questions and 1 of the last 2 questions.
- Exam due Friday, March 28 (by end of day).

The test is not meant to be confusing. Any extra information contained in the questions is designed to clarify and not obscure, and the questions are not designed to trick you. Please let me know if anything is unclear.

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Solve any 2 of the following 3 problems:

1. In multivariable calculus, given a function $f : \mathbb{R}^n \to \mathbb{R}$, you define the gradient vector ∇f by

$$\nabla f = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

In other words, it is vector field with component functions $\dfrac{\partial f}{\partial x^i}$.

Consider the more general situation where M is a smooth manifold, and $f : M \to \mathbb{R}$ a smooth function.

a. Show that the functions $\dfrac{\partial f}{\partial x^i}$ (where x is a local coordinate system) do not form the component functions of a vector field. Show, however, that they do naturally form the component functions of a 1-form.

b. Suppose we have a tensor $g \in C^\infty(M, TM^* \otimes TM^*)$, and further suppose that at each point $x \in M$, the tensor g_x is symmetric and non-degenerate. In other words,

$$g_x(v_1, v_2) = g_x(v_2, v_1),$$

and the rewritten

$$g_x : T_x M^{**} \to T_x M^*$$

is invertible. Then, **show** that g can be used to change the components $\frac{\partial f}{\partial x^i}$ into components of a vector field.

2. Consider the polar coordinates on (subsets of) $\mathbb{R}^2$ defined by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Compute $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in polar coordinates .

3. In an ODE class, you often encounter the following spring equation:

$$\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt},$$

where $x(t)$ is the displacement of a mass at time t, and k, c are constants related to the strength of the spring force and resistance force, respectively.

a. Let $y = \dfrac{dx}{dt}$, and rewrite the above equation in terms of a vector field on $M = \mathbb{R}^2$.

b. Show that flow induced by the vector field in (a) is a 1-parameter subgroup of $Gl(2, \mathbb{R})$.

Solve 1 of the following 2 problems:

4. Implicit differentiation is a useful tool in calculus. Essentially, if we have

$$f(x, y) = c,$$

then we solve for $\frac{dy}{dx}$ by implicitly differentiating and obtaining

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Prove that this is mathematically rigorous.

5. Suppose M and N are connected, oriented manifolds and $f : M \to N$ is a local diffeomorphism. Show that if f_{*p} preserves orientation at some $p \in M$, then f_* preserves orientation at all points $p \in M$.