

HW Due Thursday 4/24

1. Let  $G$  be a Lie group. A form  $\omega \in \Omega^k(G)$  is (left) invariant if

$$L_g^* \omega = \omega \quad \forall g \in G.$$

(Above,  $L_g$  is left-multiplication by  $g$ .)

- a. Show that the restriction of

$$0 \rightarrow C^\infty(G) \xrightarrow{d} \Omega^1(G) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(G) \rightarrow 0$$

to the subspaces of left-invariant forms is also a cochain complex. In other words, show that the exterior derivative of an invariant form is also invariant.

- b. If  $\mathfrak{g}$  is the Lie algebra of left-invariant vector fields, show that the cochain complex of left-invariant forms is the finite-dimensional complex

$$0 \rightarrow \mathbb{R} \xrightarrow{d} \mathfrak{g}^* \xrightarrow{d} \Lambda^2 \mathfrak{g}^* \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n \mathfrak{g}^* \rightarrow 0$$

Use the invariant description of  $d$  to give a description of  $d$  above using only the Lie algebra structure.

- c. Consider the cohomology  $H^*(\mathfrak{g}, d)$  of the finite-dimensional complex from part (b) (this is called Lie algebra cohomology). Prove that

$$H^1(\mathfrak{g}) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*.$$

What is  $H^1(\mathfrak{gl}(n, \mathbb{R}))$ , where  $\mathfrak{gl}(n, \mathbb{R})$  denotes the Lie algebra of all  $n \times n$  real matrices?