Math 531 Final
May 15, 2008

Solve any 2 of the following 3 problems:

1. Show directly that $\mathbb{R}P^n$ is a smooth manifold by giving local coordinate patches and calculating the transition functions (Hint: use homogeneous coordinates).

2. Let $M = \mathbb{R}^3$, and consider the 2-dimensional sub-bundle of $T\mathbb{R}^3$ spanned by the vector fields

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + f(x,y)\frac{\partial}{\partial z} \text{ and } \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + g(x,y)\frac{\partial}{\partial z}.$$

For what functions f, g will this be an integrable distribution?

3. Let $M = \mathbb{R}^2$. Apply Stokes Theorem to integrals of 1-forms (in local coordinates) to obtain two standard results from multi-variable calculus: The Fundamental Theorem of Line Integrals and Green's Theorem. (You don't need to remember what these theorems are to answer this question.)

Solve any 3 of the following 4 problems: (In the following, $H^k(M)$ denotes de Rham cohomology.)

4. Let M be a connected manifold. Show that $\pi_1(M) = 0$ implies that $H^1(M) = 0$. (Hint: Show that the integral of a closed 1-form over a closed 1-manifold is 0.)

5. Let M^n be a compact, connected, oriented n-manifold. Let D^n be a closed disc around a point $x_0 \in M$. Calculate the de Rham cohomology of $M^n - D^n$.

6. Let M, N be connected, compact, oriented n-manifolds. Prove that if $H^k(N) \neq 0$ and $H^k(M) = 0$ for some $k \neq 0$, then any map $f : M \to N$ must have degree 0. (Hint: Use Poincare duality.)

7. Let M be a compact, connected, oriented, n-manifold. What is the Euler class of the vector bundle $\Lambda^n TM^*$. Specifically, in what cohomology group does it live, and is it non-zero?

Solve the next problem:

8. Make up a good exam question and answer it.