

1. $y = 3e^{-2x}$. Thus, $y' = -6e^{-2x}$.

$$y' + 2y = -6e^{-2x} + 2 \cdot 3e^{-2x} = 0.$$

2. $\frac{dv}{dt} = k(250 - v)$. (where $k > 0$ for physical reasons. If this were not so, the car would be able to accelerate to an indefinitely large velocity). Equilibrium solutions occur when $0 = k(250 - v)$, and hence $v = 250$. By plugging in values of v to dv/dt , we see that $dv/dt > 0$ when $v < 250$ and $dv/dt < 0$ when $v > 250$. Therefore, the equilibrium solution is stable. This means that as time increases, the car's velocity approaches 250 km/h.

3. This is a separable equation.

$$\begin{aligned}\int \frac{dy}{y^2 + 1} &= \int 3x^2 dx \\ \tan^{-1}(y) &= x^3 + C \\ y &= \tan(x^3 + C)\end{aligned}$$

$$1 = \tan(0 + C)$$

$$C = \pi/4$$

$$y(x) = \tan(x^3 + \frac{\pi}{4})$$

Of course, because of the periodicity of \tan , one could have picked other values for C , such as $\pi/4 + 2\pi$.

4. This is a linear equation. To find the proper integrating factor, we should first put it in standard form.

$$y' + \frac{1}{2x}y = 5x^{-1/2}.$$

Then, multiply both sides by the integrating factor

$$e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2} \ln x} = e^{\ln x^{1/2}} = x^{\frac{1}{2}}.$$

Therefore, we have that

$$\begin{aligned}x^{1/2}y' + \frac{1}{2}x^{-1/2}y &= 5 \\ \frac{d}{dx}(x^{1/2}y) &= \frac{d}{dx}(5x) \\ x^{1/2}y &= 5x + C \\ y &= 5\sqrt{x} + \frac{C}{\sqrt{x}}\end{aligned}$$

5. This equation is homogeneous, and we use the substitution

$$u = y/x, \quad y = ux, \quad \frac{dy}{dx} = \frac{du}{dx}x + u.$$

It is usually a good idea (though not necessary) to first divide by x^n where n is the total degree of each term that appears.

$$\begin{aligned} x^2 \frac{dy}{dx} &= xy + y^2 \\ \frac{dy}{dx} &= \frac{y}{x} + \left(\frac{y}{x}\right)^2 \\ \frac{du}{dx}x + u &= u + u^2 \\ \frac{du}{dx}x &= u^2 \\ \frac{du}{u^2} &= \frac{dx}{x} \\ -\frac{1}{u} &= \ln|x| + C \\ -\frac{x}{y} &= \ln|x| + C \\ y &= \frac{-x}{\ln|x| + C} \end{aligned}$$

6. We use substitution

$$u = e^y, \quad u' = e^y y'.$$

Substituting gives us a linear equation

$$\begin{aligned} xe^y y' &= 2(e^y + x^3 e^{2x}) \\ xu' &= 2u + 2x^3 e^{2x} \\ u' - \frac{2}{x}u &= 2x^2 e^{2x} \\ x^{-2}(u' - 2x^{-1}u) &= x^{-2}2x^2 e^{2x} \\ \frac{d}{dx}(x^{-2}u) &= \frac{d}{dx}\left(\int 2e^{2x} dx\right) \\ x^{-2}u &= e^{2x} + C \\ x^{-2}e^y &= e^{2x} + C \\ y &= \ln(x^2 e^{2x} + Cx^2) \end{aligned}$$

7. We first check exactness by showing that

$$\frac{\partial}{\partial y}\left(x^3 + \frac{y}{x}\right) = \frac{1}{x} = \frac{\partial}{\partial x}(y^2 + \ln x).$$

Therefore, this equation is equivalent to $\frac{d}{dx}F(x, y) = 0$. Solving for F , we find that

$$F(x, y) = \int (x^3 + \frac{y}{x})dx = \frac{1}{4}x^4 + y \ln x + g(y).$$

$$y^2 + \ln x = \frac{\partial F}{\partial y} = \ln x + g'(y)$$

$$g'(y) = y^2$$

$$g(y) = \frac{1}{3}y^3$$

Therefore, the final solution is $F = C$, which is

$$\frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 = C.$$

8. Since there is no x dependence in the equation, we use the substitution

$$v = \frac{dy}{dx}, \quad d^2y dx^2 = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = \frac{dv}{dy} v.$$

Substituting, we get a (linear) first-order equation

$$yy'' + (y')^2 = yy'$$

$$y \frac{dv}{dy} v + v^2 = yv$$

$$\frac{dv}{dy} + \frac{1}{y}v = 1$$

9. We have a separable equation

$$\frac{dx}{7x(x-13)} = dt$$

which we will solve by partial fractions.

$$\frac{1}{7x(x-13)} = \frac{A}{7x} + \frac{B}{x-13}$$

$$A(x-13) + B(7x) = 1$$

$$-13A + 0 = 1$$

$$(A + 7B)x = 0x$$

$$A = \frac{-1}{13}, \quad B = -\frac{A}{7} = \frac{1}{13 \cdot 7}$$

Using this, we can now perform in the integral

$$\begin{aligned} \int \frac{dx}{7x(x-13)} &= \int dt \\ \int \frac{1}{7 \cdot 13} \left(\frac{-1}{x} + \frac{1}{x-13} \right) &= \int dt \\ \frac{1}{91} (-\ln|x| + \ln|x-13|) &= t + c \\ \ln \left| \frac{x-13}{x} \right| &= 91t + c \\ \frac{x-13}{x} &= Ce^{91t} \\ x &= \frac{13}{1 - Ce^{91t}} \\ \\ 17 &= \frac{13}{1 - C} \\ C &= \frac{4}{17} \\ \\ x &= \frac{13}{1 - \frac{4}{17}e^{91t}} \\ x &= \frac{221}{17 - 4e^{91t}} \end{aligned}$$

10. $\frac{dP}{dt} = kP^2$ is a separable equation, and has solution

$$P = \frac{1}{C - kt}.$$

Letting t be the number of years after 1988, we have that $P(0) = 12$, $P(10) = 24$, and plugging these in we find that $C = \frac{1}{12}$, $k = \frac{1}{240}$, giving us

$$P = \frac{240}{20 - t}.$$

Setting $P = 48$, we find that $t = 15$, and we see that P has a vertical asymptote at $t = 20$, which means the population explodes around 2008 (which is *next year...* uh-oh....)

11. Equilibrium solutions occur when $3x - x^2 = 0$, which is $x = 0, 3$. The derivative of x is negative when $x < 0$, positive for $0 < x < 3$, and negative for $x > 3$ (this can be seen because $\frac{dx}{dt} = 3x - x^2$). Therefore, $x = 0$ is an unstable equilibrium, and $x = 3$ is a stable equilibrium.

12. $\frac{dv}{dt} = -kv$, $v(0) = 40, v(10) = 20, x(0) = x_0$. After solving the differential equation and determining the constants, we have

$$\begin{aligned}\frac{dx}{dt} &= v(t) = 40e^{-.1 \ln 2t} \\ x(t) &= \frac{-40}{.1 \ln 2} e^{-.1 \ln 2t} + C = \frac{-400}{\ln 2} e^{-.1 \ln 2t} + C \\ x(t) &= \frac{400}{\ln 2} (1 - e^{-.1 \ln 2t}) + x_0\end{aligned}$$

Therefore, $\lim_{t \rightarrow \infty} x(t) = \frac{400}{\ln 2} + x_0$, so the boat coasts for $\frac{400}{\ln 2}$ ft (about 577 ft).

13. Using Euler's method with step-size .1, we get $y(.2) \approx .81$.

| i | x_i | y_i | $y_{i+1} = -.1y_i + y_i$ |
|-----|-------|-------|-------------------------------------|
| 0 | 0 | 1 | $1 - .1 = .9$ |
| 1 | .1 | .9 | $.9 - .1 \cdot .9 = .9 - .09 = .81$ |
| 2 | .2 | .81 | |