1. (10 points) Show that $y = x^4 e^x$ is a solution to the differential equation

$$y' = \left(\frac{4}{x} + 1\right) y.$$

$y' = 4x^3 e^x + x^4 e^x$
$(\frac{4}{x} + 1)y = \frac{4}{x} x^4 e^x + x^4 e^x = 4x^3 e^x + x^4 e^x$.
Therefore, we have a y such that $y' = (4/x + 1)y$.

2. (10 points) Find the general solution to

$$\frac{dy}{dx} + 2xy = x.$$

This is a linear equation, so we multiply by the integrating factor

$$e^{\int 2x dx} = e^{x^2},$$

resulting in

$$\begin{aligned} e^{x^2} \frac{dy}{dx} + e^{x^2} 2xy &= e^{x^2} x \\ \frac{d}{dx}\left(e^{x^2} y\right) &= \frac{d}{dx}\left(\int e^{x^2} x dx\right) = \frac{d}{dx}\left(\frac{1}{2} e^{x^2}\right) \\ e^{x^2} y &= \frac{1}{2} e^{x^2} + C \\ y &= \frac{1}{2} + C e^{-x^2} \end{aligned}$$

3. Newton's law of cooling states that the time rate of change of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium.

a. (10 points) Write a differential equation expressing Newton's law of cooling.

$$\frac{dT}{dt} = k(A - T) = -k(T - A), \qquad (k > 0).$$

Note the sign of k for physical reasons. If an object is cooler than the surrounding temperature, the temperature will increase. Hence, $\frac{dT}{dt} > 0$ if $T < A$.

b. (10 points) Find all equilibrium solutions and determine their stability. Setting $0 = \frac{dT}{dt} = k(A - T)$ gives us $T = A$ (note that k is a constant here, not a variable depending on t, and hence $k = 0$ is not considered an equilibrium solution). To check the stability, we see that $\frac{dT}{dt} > 0$ for $T < A$ and $\frac{dT}{dt} < 0$ for $T > A$. This implies that $T = A$ is a stable equilibrium (draw a phase diagram).

c. (5 points) What do the equilibrium solution(s) in part (b) tell us about the temperature of the object?
As time increases, the temperature of the object will tend towards the temperature A of the surrounding medium, regardless of what the starting temperature of the object was.

4. (25 points) Suppose an object slides along the ground, and the only force acting on it is friction. Let $x(t)$ be the horizontal displacement in feet of the object after t seconds, and suppose that $v = \frac{dx}{dt}$ satisfies the equation

$$\frac{dv}{dt} = -kv.$$

The object's velocity is 25 ft/s after 0 seconds, and the velocity is $25e^{-2}$ ft/s after 1 seconds.

a. What is the velocity at time t?

$$\begin{aligned} \frac{dv}{dt} &= -kv \\ \int \frac{dv}{v} &= \int -kdt \\ \ln|v| &= -kt + c \\ v &= Ce^{-kt} \end{aligned}$$

Plugging in our two data points for velocity gives us $v(0) = 25 = C$ and $25e^{-2} = v(1) = 25e^{-k}$, hence $k = 2$ and

$$v(t) = 25e^{-2t}.$$

b. How far has the object travelled after t seconds?

$$\begin{aligned} \frac{dx}{dt} &= 25e^{-2t} \\ x &= \int 25e^{-2t}dt = -\frac{25}{2}e^{-2t} + C \end{aligned}$$

$$x_0 = -\frac{25}{2}e^0 + C$$

Since we are considering x as displacement, this means we set $x_0 = 0$ and have

$$x(t) = -\frac{25}{2}e^{-2t} + \frac{25}{2}.$$

c. After a large amount of time, approximately how far will the object have travelled?

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \left(-\frac{25}{2}e^{-2t} + \frac{25}{2}\right) = 0 + \frac{25}{2}.$$

Hence, for large values of t, the distance travelled is approximately $\frac{25}{2}$ ft.

5. (10 points) Solve the initial value problem

$$2xyy' = x^2 + 2y^2, \quad y(1) = 2, \quad (x > 0).$$

This is a homogeneous equation (you can also use the substitution of $u = y^2$ and obtain a linear equation). Using

$$u = \frac{y}{x}, \quad y = ux, \quad \frac{dy}{dx} = \frac{du}{dx}x + u,$$

we get a separable equation by

$$\begin{aligned}
2\frac{y}{x}\frac{dy}{dx} &= 1 + 2\left(\frac{y}{x}\right)^2 \\
2u(\frac{du}{dx}x + u) &= 1 + 2u^2 \\
2xu\frac{du}{dx} &= 1 \\
\int 2udu &= \int \frac{dx}{x} \\
= \frac{y^2}{x^2} = u^2 &= \ln x + C \\
y^2 &= x^2(\ln x + C) \\
\\
2^2 &= 1^2(\ln 1 + C) = C \\
\\
y^2 &= x^2(\ln x + 4) \\
y &= x\sqrt{\ln x + 4}
\end{aligned}$$

Notice that because of our initial condition, we see that $y > 0$ and hence we take the positive square root.

6. (10 points) Show that

$$\left(2x + e^{x^2y}2xy\right) dx + \left(e^{x^2y}x^2 + \cos y\right) dy = 0$$

is exact, and find a general solution.

To check exactness, show that

$$\frac{\partial}{\partial y}\left(2x + e^{x^2y}2xy\right) = \frac{\partial}{\partial x}\left(e^{x^2y}x^2 + \cos y\right),$$

both of which are equal to $e^{x^2y}2x^3y + e^{x^2y}2x$. Since the equation is exact, we will obtain a solution of the form $F(x,y) = C$ where

$$F = \int (2x + e^{x^2y}2xy)dx + g(y) = x^2 + e^{x^2y} + g(y)$$

$$\begin{aligned} \frac{\partial F}{\partial y} &= e^{x^2y}x^2 + \cos y \\ g'(y) &= \cos y \\ g(y) &= \sin y \end{aligned}$$

$$F(x,y) = x^2 + e^{x^2y} + \sin y = C.$$

7. (10 points) The motion of a mass m on a spring is governed by Hooke's Law, which states that the restoring force of the spring is proportional to the displacement x of the mass from its equilibrium position. If no other forces act, the motion of the mass is governed by the second-order differential equation

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right) x = 0,$$

(where $k > 0$ is a constant). Using substitution, reduce this second-order equation to a first-order equation. You do not need to solve the equation.

Use the substitution

$$p = \frac{dx}{dt}, \quad d^2xdt^2 = \frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt} = \frac{dp}{dx}p,$$

to give the separable equation

$$\frac{dp}{dx}p + \left(\frac{k}{m}\right) x = 0.$$