Math 303 Fall 2007 Midterm 2 Review Answers

1. (3.1: 20-26, 3.2: 1-12) Show whether the following sets of sets of functions are linearly dependent or independent:
a. $\{\cos 2x, \sin 2x\}$
The Wronskian

$$W = W(\cos 2x, \sin 2x) = \begin{vmatrix} \cos 2x & \sin 2x \\ (\cos 2x)' & (\sin 2x)' \end{vmatrix}$$
$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 1$$

Since $W \neq 0$, the functions $\cos 2x$ and $\sin 2x$ are linearly independent.

b. $\{\cos^2 x, \sin^2 x\}$

$$W = \begin{vmatrix} \cos^2 x & \sin^2 x \\ -2\cos x \sin x & 2\sin x \cos x \end{vmatrix} = 2\sin x \cos x$$

The Wronskian W is not identically 0 (plug in for a sample value of x like $\pi/4$), and therefore the pair of functions is linearly independent.

c. $\{\cos^2 x, \sin^2 x, 4\}$.
The relation

$$4\cos^2 x + 4\sin^2 x + (-1) \cdot 4 = 0$$

shows the linear dependence of the three given functions.

2. (3.1: 1-16, 3.2: 13-20) Verify that the following functions are solutions to the given differential equation. Solve the initial value problem. (Bonus: What does the Wronskian of your solutions have to do with solving the initial value problem?)
a. $x^2y'' + 2xy' - 6y = 0; y_1 = x^2, y_2 = x^{-3}; y(1) = 3, y'(1) = 1$

$$x^2y_1'' + 2xy_1' - 6y = x^2 2 + 2x2x - 6x^2 = 0.$$
$$x^2y_2'' + 2xy_2' - 6y = x^2(12x^{-5}) + 2x(-3x^4) - 6x^{-3} = 0$$

$$\begin{cases} c_1y_1(1) + c_2y_2(1) = 3 \\ c_1y_1'(1) + c_2y_2'(1) = 1 \end{cases} \qquad \begin{cases} c_1 + c_2 = 3 \\ 2c_1 - 3c_2 = 1 \end{cases} \quad \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$$

Therefore, $y = c_1y_1 + c_2y_2 = 2x^2 + x^{-3}$.

b. $y^{(3)} - 6y'' + 11y' - 6y = 0; y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x};$
$y(0) = 0, y'(0) = 1, y''(0) = 3.$

Verifying that y_1, y_2, y_3 are solutions is done by plugging in the functions and obtaining. For y_3 this looks like

$$e^{3x}(3^3 - 6 \cdot 3^2 + 1 \cdot 3 - 6 \cdot 3) = 0.$$

The solution $y = -e^x + e^{2x}$ is obtained by solving

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 3c_3 = 1 \\ c_1 + 4c_2 + 9c_3 = 3 \end{cases}$$

3. (3.1: 33-48, 3.3: 1-20, 33-36) Find general solutions to the following homogeneous equations.
In each of the following, we first solve the characteristic equation and use the roots to construct the general solution.
a. $3y' - y = 0$
$3r - 1 = 0$. $r = \frac{1}{3}$. $y = c_1e^{\frac{1}{3}x}$.
b. $y'' + 2y' - 15y = 0$
$r^2 + 2r - 15 = 0$. $r = -5, 3$. $y = c_1e^{-5x} + c_xe^{3x}$.
c. $9y^{(3)} + 12y'' + 4y' = 0$
$9r^3 + 12r^2 + 4r = r(3r + 2)^2$. $y = c_1 + (c_2 + c_3x)e^{-2/3x}$.
d. $y^{(4)} - 8y'' + 16y = 0$
$r^4 - 8r^2 + 16 = (r^2 - 4)^2 = 0$. $y = (c_1 + c_2x)e^{-2x} + (c_3 + c_4x)e^{2x}$.

4. (3.1: 1-16, 3.3: 21-26) Solve the following initial value problems.
We found the general solutions in question 3.
a. $3y' - y = 0; y(0) = 5$
$y = 5e^{1/3x}$.
b. $y'' + 2y' - 15y = 0; y(0) = 0, y'(0) = 4$
$y = \frac{1}{2}e^{-5x} - \frac{1}{2}e^{3x}$.
c. $9y^{(3)} + 12y'' + 4y' = 0; y(0) = 0, y'(0) = 1, y''(0) = \frac{10}{3}$
$y = \frac{21}{2} + (\frac{-21}{2} + 8x)e^{-2/3x}$.

5. Show that the function $y = \sin x$ satisfies the equation

$$yy'' - (y')^2 = -1.$$

Does it then follow that $y = 3\sin x$ is also a solution? Why or why not?
The original differential equation is **not linear**, so it does not follow that $y = 3\sin x$ is a solution. In fact, it is easily verified that $y = 3\sin x$ is not a solution.

6. (3.2: 21-24, 3.5: 1-20, 31-40) Find a general solution to the following nonhomogeneous equations and/or solve the initial value problem.
a. $y'' + y = 3x; y(0) = 2, y'(0) = -2$
First, we solve the complimentary homogeneous equation $y'' + y = 0$, giving us $y_c = c_1 \cos x + c_2 \sin x$. Then, we guess that a particular solution of the non-homogeneous equation will be of the form $A3x$ plus terms coming from the derivatives (a constant). Because none of these appear in the solution to the homogeneous equation, we make the guess $y_p = Ax + B$ and plugging in find $y_p = 3x$.

b. $y'' + 9y = 2\cos 3x + 3\sin 3x$
First, $y_c = c_1 \cos 3x + c_2 \sin 3x$. For our guess at y_p, we take the generalized right hand side plus derivatives, which will look like $A\cos 3x + B\sin 3x$ and multiply it by the lowest power of x so that it does not appear in the complimentary solution. This gives $y_p = Ax\cos 3x + Bx\sin 3x$, and after plugging in, we solve

$$-6A\sin 3x + 6B\cos 3x = 2\cos 3x + 3\sin 3x$$

to give us $y_p = -\frac{1}{2}x\cos 3x + \frac{1}{3}\sin 3x$, and hence $y = y_p + y_c = -\frac{1}{2}x\cos 3x + \frac{1}{3}\sin 3x + c_1\cos 3x + c_2\sin 3x$.

7. (3.5:21-30) Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.
a. $y'' - 2y' + 2y = e^x \sin x$
First, $y_c = e^x(c_1\cos x + c_2\sin x)$. Initially, we would guess

$$y_i = Ae^x \sin x + Be^x \cos x,$$

but we notice that that is a solution to the complimentary equation, so we multiply by a power of x to eliminate such duplication, giving us

$$y_p = xe^x(A\sin x + B\cos x).$$

8. (3.4: 24-33) Show that if a mass-spring-dashpot system with no external force is underdamped (i.e. $c^2 < 4km$), then the mass passes through the equilibrium position an infinite number of times.
Solving $mx'' + cx' + kx = 0$ with $c^2 < 4mk$ gives a solution of the form

$$x(t) = e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

Finding when the mass passes through the equilibrium position is equivalent to solving $x = 0$, and $c_1 \cos \omega_1 t + c_2 \sin \omega_1 t = 0$ for an infinite number of $t > 0$.

9. (3.4: 15-21) Find the position function for the following mass-spring-dashpot system. What happens to the position for large time?
a. $m = \frac{1}{2}, c = 3, k = 4; x_0 = 2, v_0 = 0$
$\frac{1}{2}x'' + 3x' + 4x = 0$. $x(t) = c_1 e^{-2t} + c_2 e^{-4t}$.

$$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 - 4c_2 = 0 \end{cases} \qquad \begin{cases} c_1 = 4 \\ c_2 = -2 \end{cases}$$

$x(t) = 4e^{-2t} - 2e^{-4t}$, and $\lim_{t\to\infty} x(t) = 0$, so the mass' position is approximately at equilibrium for large time.

10. (3.4: 1-4) What is the amplitude and period for the undamped mass-spring system with $m = 2, k = 8$ and $x_0 = 3, v_0 = 8$?
$2x'' + 8x = 0$. $x(t) = A \cos 2t + B \sin 2t$. $x(t) = 3 \cos 2t + 4 \sin 2t$. Written like this, we see that the amplitude will be $\sqrt{3^2 + 4^2} = 5$, and the period will be $\frac{2\pi}{2} = \pi$. This follows explicitly from

$$\begin{aligned} x &= 3 \cos 2t + 4 \sin 2t \\ &= 5(\frac{3}{5} \cos 2t + \frac{4}{5} \sin 2t) \\ &= 5 \cos(2t - \alpha) \end{aligned}$$

where $\cos \alpha = \frac{3}{5}, \sin \alpha = \frac{4}{5}$. (The last step above is given by a trig identity for $\cos(a - b)$.

11. (3.6: 1-14)a. Find the position function for the undamped mass-spring system with external force $4e^{-t}$ given by the equation

$$x'' + x = 4e^{-t}; x_0 = 3, v_0 = -2.$$

$x = 2e^{-t} + c_1 \cos t + c_2 \sin t = -2e^{-t} + \cos t$.

b. Find the amplitude of the steady-periodic solution to

$$mx'' + kx = F_0 \cos(\omega t).$$

Assuming $\omega \neq \sqrt{\frac{k}{m}}$, we find

$$x_sp(t) = x_p(t) = \frac{F_0}{k - m\omega^2}\cos\omega t = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t.$$

Therefore, the amplitude is the coefficient $\frac{F_0/m}{\omega_0^2-\omega^2}$.

12. (3.6: 15-18) In the following mass-spring-dashpot systems with external force $F_0\cos(\omega t)$, is there practical resonance for some $\omega > 0$? If so, at what frequency ω will this occur? (Hint: The formula for the amplitude of the steady-periodic solution is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

a. $2x'' + \sqrt{2}x' + 5x = F_0\cos(\omega t)$. The amplitude will be

$$C(\omega) = \frac{F_0}{\sqrt{(5 - 2\omega^2)^2 + (\sqrt{2}\omega)^2}} = \frac{F_0}{\sqrt{4\omega^4 - 18\omega^2 + 25}}.$$

To maximize this, we take the derivative and set it equal to zero, giving us

$$\begin{aligned}
0 =& C'(\omega) = \frac{-1/2F_0}{(4\omega^4 - 18\omega^2 + 25)^{3/2}}\left(16\omega^3 - 36\omega\right)\\
0 =& 4\omega(4\omega^2 - 9)\\
\omega =& 0, -\frac{3}{2}, \frac{3}{2}
\end{aligned}$$

We normally assume $\omega > 0$ (this is convention; notice that $\cos(-\omega t) = \cos(\omega t)$), and we can see that $\omega = \frac{3}{2}$ is the global maximum, giving us the value where practical resonance occurs.

13. Transform the mass-spring-dashpot equation $mx'' + cx' + kx = 0$ into a system of first-order equations.
Let $y = x'$. Then

$$\begin{cases} y = x' \\ my' + cy + kx = 0 \end{cases}$$