Math 303 Fall 2007 Final Exam Review Answers

1. These can all be seen by directly substituting the solution into the differential equation. For instance, for part (b), we have that

$$\vec{x}' = \lambda \vec{v} e^{\lambda t}, \quad A\vec{x} = (A\vec{v})e^{\lambda t}.$$

Therefore (since $e^{\lambda t} \neq 0$), $\vec{x}(t) = \vec{v} e^{\lambda t}$ is a solution if and only if $A\vec{v} = \lambda \vec{v}$, which is the definition for an eigenvector and eigenvalue of A.

2. $x(t) = c_1 e^{-2t} + c_2 e^{-3t}$. Notice that the characteristic equation from part (a) is the same polynomial as the eigenvalue equation in part (b). In part (b), you use $x' = y$ and solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \mathbf{x},$$

which gives you the solution

$$\mathbf{x}(t) = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-3t} \\ -2c_1 e^{-2t} - 3c_2 e^{-3t} \end{bmatrix}.$$

The top row is the solution to the original equation, and the second row ($y(t)$) is $\frac{dx}{dt}$).

3. **I meant for this problem to be

$$\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}.$$

Notice the negative sign in front of the 3 (it makes things much nicer). However, the answer to the problem, as stated, is....

$$\mathbf{x_1} = e^{3t} \begin{bmatrix} \cos\sqrt{5}t \\ \sqrt{5}\sin\sqrt{5}t \end{bmatrix}, \mathbf{x_2} = e^{3t} \begin{bmatrix} \sin\sqrt{5}t \\ -\sqrt{5}\cos\sqrt{5}t \end{bmatrix}.$$

Linear independence is shown using the Wronskian,

$$W(\mathbf{x_1}, \mathbf{x_2}) = e^{3t} \begin{vmatrix} \cos\sqrt{5}t & \sin\sqrt{5}t \\ \sqrt{5}\sin\sqrt{5}t & -\sqrt{5}\cos\sqrt{5}t \end{vmatrix} = \sqrt{5}e^{3t}$$

4. The general solution

$$\overrightarrow{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

is worked out in Example 1 of 5.2. The solution to the initial value problem is

$$\begin{cases} x(t) = e^{-2t} + 2e^{5t} \\ y(t) = -3e^{-2t} + e^{5t} \end{cases} .$$

5. Please note that there are multiple equivalent ways of writing solutions on these problems (depending on the basis vectors one chooses for eigenspaces).

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \Rightarrow \overrightarrow{x}(t) = e^{4t} \begin{bmatrix} c_1 \cos 3t - c_2 \sin 3t \\ c_1 \sin 3t + c_2 \cos 3t \end{bmatrix} \quad \text{(5.2 Ex.3)}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \Rightarrow \Phi(t) = \begin{bmatrix} e^t & 2\cos 2t - \sin 2t & \cos 2t + 2\sin 2t \\ -e^t & -3\cos 2t + \sin 2t & \cos 2t - 3\sin 2t \\ & 3\cos 2t + \sin 2t & 3\sin 2t - \cos 2t \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \overrightarrow{x}(t) = c_1 \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -2t+1 \\ -2t-5 \\ 2t+1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -t^2+t+1 \\ -t^2-5t \\ t^2+t \end{bmatrix} e^{-t}$$

(5.4 Ex. 4)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \Phi(t) = \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ e^{2t} & 0 & e^{9t} \\ 0 & e^{2t} & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} \Rightarrow \overrightarrow{x}(t) = \begin{bmatrix} -c_1 - 2c_2 + c_3 \\ c_2 + c_3 t \\ c_1 - 2c_3 t \end{bmatrix} e^t$$

6.

$$e^{\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} t} = e^{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t} e^{\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

$$e^{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} t} = \begin{bmatrix} e^t & 0 & 2te^t \\ 0 & e^t & -3te^t \\ 0 & 0 & e^t \end{bmatrix}$$

7. Letting $x_1(t), x_2(t)$ be the amount of salt in tanks 1 and 2, respectively, we obtain the system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = -\frac{10}{50} x_1 \\ \frac{dx_2}{dt} = -\frac{10}{50} x_1 - \frac{10}{25} x_2 \end{cases}$$

Solving this, we obtain

$$\mathbf{x} = \begin{bmatrix} c_1 e^{-1/5t} \\ c_1 e^{-1/5t} + c_2 e^{-2/5t} \end{bmatrix} = \begin{bmatrix} 15e^{-1/5t} \\ 15e^{-1/5t} - 15e^{-2/5t} \end{bmatrix}$$

8.

$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = x - 2y \end{cases}$ is linear, and has only one equilibrium solution, located at $(0, 0)$.

The eigenvalues of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $\lambda = -1, -3$, which are both negative, so the equilibrium is stable (in fact, it is an asymptotically stable nodal sink).

$\begin{cases} \frac{dx}{dt} = 1 - y^2 \\ \frac{dy}{dt} = x + 2y \end{cases}$ has two equilibrium solutions, located at $(-2, 1)$ and $(2, -1)$.

The Jacobean at (-2,1) is $\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$ and has eigenvalues $\lambda = 1 \pm i$. The real parts of the eigenvalue are positive, so the equilibrium solution is unstable (in fact, it is an unstable spiral source). The Jacobean at (1,-1) is $\begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$ with eigenvalues $\lambda = 1 \pm \sqrt{3}$, one of which is positive, and the other negative. Therefore, that equilibrium point is also unstable (and in fact, is an unstable saddle point).