Math 205: Midterm 2 Solutions

1. (15 points) Let $f(x,y) = x^{3/2}\sin y$.
   a. Find the 2nd-order Taylor polynomial for $f(x,y)$ at the point $(1,0)$.
   b. Use part (a) to approximate $\sqrt{1.1^3}\sin(-.1)$

$$\begin{aligned}
p_2(x,y) =& f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\
&+ f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\
=& y + \frac{3}{2}(x-1)y \\
\sqrt{1.1^3}\sin(-.1) =& f(1.1,-.1) \approx p_2(1.1,-.1) \\
\sqrt{1.1^3}\sin(-.1) \approx& -.1 + \frac{3}{2}(1.1-1)(-.1) = -.115
\end{aligned}$$

2. (20 points) Let $f(x,y) = x^3y + 12x^2 - 8y$. Find all critical points of f and classify their behavior (i.e. determine if a critical point is a local minimum, maximum, saddle, or if it cannot be determined).

$$\begin{cases} 0 = \frac{\partial f}{\partial x} = 3x^2y + 24x \\ 0 = \frac{\partial f}{\partial y} = 12y + 48 \end{cases} \Rightarrow x = 2, y = -4.$$

$$H(x,y) = \begin{bmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{bmatrix}$$

$$H(2,-4) = \begin{bmatrix} -24 & 12 \\ 12 & 0 \end{bmatrix}$$

$$Det(H(2,-4)) < 0 \Rightarrow \text{ (2,-4) is a saddle point.}$$

3. (20 points) Consider the function $f(x,y) = x^2y$. Find the absolute maximum and minimum values of f on the bounded region

$$x^2 + 2y^2 \leq 16.$$

First, we find critical points in the interior region:

$$\nabla f = \mathbf{0}$$

$$\begin{cases} 0 = f_x = 2xy \\ 0 = f_y = x^2 \end{cases} \Rightarrow x = 0.$$

$$f(0,y) = 0^2y = 0$$

Therefore, any point $(0, y)$ (for $y^2 < 8$) will be a critical point, and $f(0, y) = 0$ at any such point.

We now check for extremal points on the boundary $g(x, y) = x^2 + 2y^2 = 16$ using Lagrange multipliers. Note that we may assume $x \neq 0$ since we have already checked those points.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 16 \end{cases}$$

$$\begin{cases} 2xy = \lambda 2x \\ x^2 = \lambda 4y \\ x^2 + 2y^2 = 16 \end{cases}$$

$$\Rightarrow y^2 = \frac{8}{3}, y = \pm\sqrt{8/3}, x^2 = \frac{32}{3}, x = \pm 4\sqrt{\frac{2}{3}}.$$

The above calculation gives us 4 points, and we must check the value of f at each of these and compare with the other values. Doing so shows us that

$$f(\pm 4\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}) = \frac{64}{3}\sqrt{\frac{2}{3}} \text{ is the maximum.}$$

$$f(\pm 4\sqrt{\frac{2}{3}}, -2\sqrt{\frac{2}{3}} = -\frac{64}{3}\sqrt{\frac{2}{3}} \text{ is the minimum.}$$

4. (10 points) *In the following problem, you do not have to solve for a final answer. You only have to translate the question into solving a system of algebraic equations.*

The base of an open-top aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, set up a problem to find the dimensions of the aquarium that minimize the cost of the materials. Be sure to

a. Draw a picture.
b. Identify and label your variable names.
c. Take any necessary derivatives.
d. Note what your system of equations will tell you.

The setup for this problem depends on the picture drawn. In here, assume we have a rectangular tank with width and depth x and y and height z. Then, we wish to maximize the total cost of material subject to the constraint of a fixed volume. The cost function will be

$$C(x, y, z) = 5xy + 2xz + 2yz,$$

(technically, C should be some constant times the above equation. However, that constant is irrelevant when maximizing.) subject to the constraint

$$xyz = V$$

(where V is a constant). To optimize such a constrained system, we use Lagrange multipliers, giving us the system of equations

$$\begin{cases} 5y + 2z = \lambda yz \\ 5x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz = V \end{cases}$$

This is a system of 4 equations with 4 unknowns. The solution(s) (x, y, z, λ) will give us the dimensions (x, y, z) which optimize C subject to the constraint $xyz = V$.

5. (10 points) Find the volume of the 3-dimensional region bounded by the planes

$$x = 0, x = 1, y = 0, y = 2, z = 0, z = 3x + 4y.$$

$$\int_0^1 \int_0^2 (3x + 4y) dy dx = \int_0^1 3xy + 2y^2|_0^2 dx = \int_0^1 (6x + 8) dx$$
$$= 3x^2 + 8x|_0^1 = 11$$

6. (10 points) Compute the integral

$$\int\int_D x^3 y^2 dA$$

where $D = \{(x, y) \mid 0 \leq x \leq 2, -x \leq y \leq x.\}$.

$$\int\int_D x^3 y^2 dA = \int_{x=0}^{x=2} \int_{y=-x}^{y=x} x^3 y^2 dy dx = \int_0^2 \frac{2}{3} x^6 dx = \frac{2^8}{21}$$

7. (15 points) Compute the integral

$$\int_0^4 \int_{y/2}^2 \sin(x^2) dx dy$$

by
a. Drawing the region being integrated over.
b. Switching the order of integration and then integrating.

The region drawn should be the region bounded by the lines

$$x = 0, y = 0, y = 4, y = 2x.$$

This gives

$$\int_0^4 \int_{y/2}^2 \sin(x^2) dx dy = \int_{x=0}^{x=2} \int_{y=0}^{y=2x} \sin(x^2) dy dx = \int_0^2 2x \sin(x^2) dx = 1 - \cos 4.$$