Math 205: Midterm 1 Solutions

1. Let $\mathbf{a} = (0,3,-1), \mathbf{b} = (2,-1,1), \mathbf{c} = (1,1,2)$.
a. $(3\mathbf{a} + \mathbf{b}) - \mathbf{c} = 3(0,3,-1) + (2,-1,1) - (1,1,2) = (1,7,4)$.
b. $\mathbf{a} \cdot \mathbf{b} = (0,3,-1) \cdot (2,-1,1) = 0 * 2 + 3 * -1 + -1 * 1 = -4$
c. Find the angle between the two vectors $\mathbf{b}$ and $\mathbf{c}$.

$$\begin{aligned}
\mathbf{b} \cdot \mathbf{c} &= \|\mathbf{b}\|\|\mathbf{c}\| \cos\theta \\
\cos\theta &= \frac{\mathbf{b} \cdot \mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|} \\
\cos\theta &= \frac{(2,-1,1) \cdot (1,1,2)}{\sqrt{2^2+1^2+1^2}\sqrt{1^2+1^2+2^2}} = \frac{1}{2} \\
\theta &= \frac{\pi}{3}
\end{aligned}$$

d. Find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{c}$.

$$\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 7\mathbf{i} - \mathbf{j} - 3\mathbf{k}.$$

2. Find the distance between the point $(5,-1,2)$ and the plane $2x - 2y + z = 7$.
Let $\mathbf{v}$ be a vector from the point to the plane, and let $\mathbf{n}$ be the normal vector to the plane. From the coefficients of the equation for the plane, we see that

$$\mathbf{n} = (2,-2,1),$$

and if we choose $(0,0,7)$ as our point on the plane, then

$$\mathbf{v} = (5,-1,2) - (0,0,7) = (5,-1,-5).$$

The distance between the point and the plane will then be the distance of the vector $\mathbf{v}$ projected onto the normal vector $\mathbf{n}$.

$$\begin{aligned}
Proj_{\mathbf{n}}\mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} \\
&= \frac{(5,-1,5) \cdot (2,-2,1)}{(2,-2,1) \cdot (2,-2,1)}(2,-2,1) = \frac{7}{9}(2,-2,1) \\
\|Proj_{\mathbf{n}}\mathbf{v}\| &= \frac{7}{9}\sqrt{9} = \frac{7}{3}
\end{aligned}$$

The distance between the point and plane is 7/3.

3. Let $\mathbf{r}(t) = t^2\mathbf{i} - (t+2)\mathbf{j} + e^t\mathbf{k}$ be a curve in $\mathbb{R}^3$. Write an equation for the tangent line to the curve at $t = 3$.
The tangent line $\mathbf{l}(t)$ is given by the equation

$$\mathbf{l}(t) = \mathbf{r}(3) + \mathbf{r}'(3)t.$$

(It goes through the point $\mathbf{r}(3)$ and has direction $\mathbf{r}'(3)$.) We then calculate

$$\mathbf{r}(3) = 9\mathbf{i} - 5\mathbf{j} + e^3\mathbf{k}$$
$$\mathbf{r}'(t) = 2t\mathbf{i} - \mathbf{j} + e^t\mathbf{k}$$
$$\mathbf{r}'(t) = 6\mathbf{i} - \mathbf{j} + e^e\mathbf{k}$$

$$\mathbf{l}(t) = (9 + 6t)\mathbf{i} + (-5 - t)\mathbf{j} + (e^3 + e^3t)\mathbf{k}$$

4. Consider the surface given by the equation

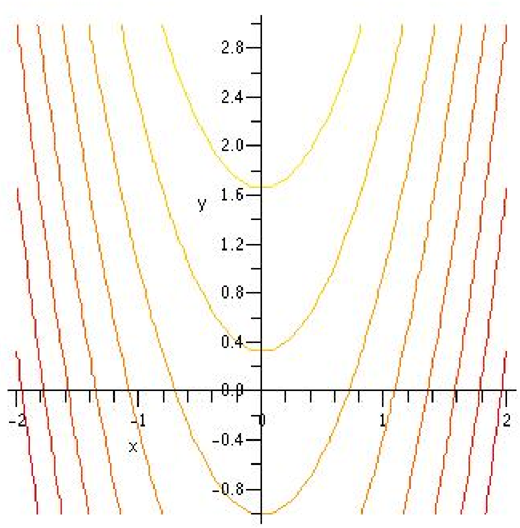
$$z = y - 2x^2.$$

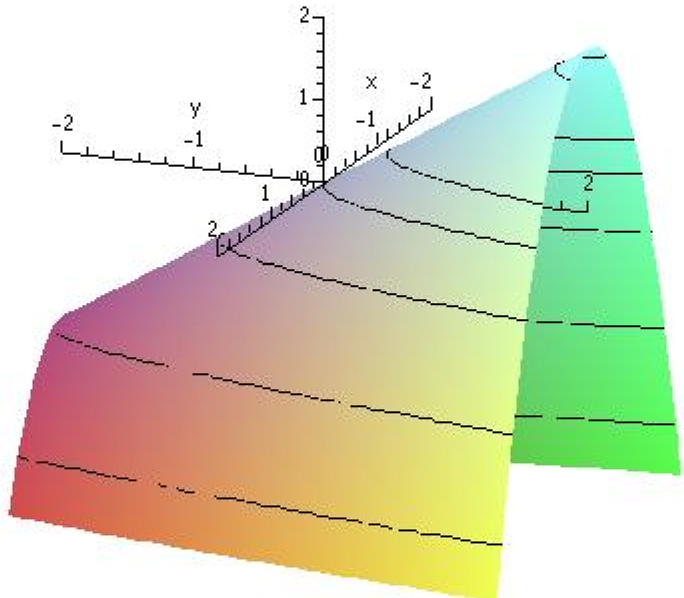
a. Find the level curve at an arbitrary height and then sketch several sample level curves (e.g. curves for $c = -1, 0, 1, 2$ is sufficient).
b. Then, use this information to give a *rough* sketch of the surface.

A level curve at height $z = c$ is given by the equation

$$c = y - 2x^2$$
$$y = 2x^2 + c$$

Thus, the level curves will all be parabolas shifted vertically by the constant c.





5. Evaluate the following limits. If they fail to exist, explain why.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + 1} = \frac{0^2 - 0^2}{0^2 + 1} = \frac{0}{1} = 0.$$

Since the numerator and denominator of the above function are polynomial, and the denominator is not 0 at the point $(0, 0)$, then the limit is obtained by plugging in the point $(x, y) = (0, 0)$.

Consider approaching $(0, 0)$ through the curve $y = mx$. Then

$$\lim_{(x,mx)\to(0,0)} \frac{x^2 - y^2}{x^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2}(1 - m^2) = 1 - m^2$$

The above limit gives a different value for different curves approaching $(0, 0)$, so the overall limit does not exist.

6. (15 points) Consider the surface given by the function

$$z = f(x, y) = (x + y)e^{2y}.$$

a. Find ∇f at the point $(2, 2)$.
b. Write an equation for the tangent plane at the point $(2, 2)$.
c. Using linear approximation, estimate $f(1.8, 2.1)$.

a. $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (e^{2y}, (2x + 2y + 1)e^{2y})$.

$$\nabla f(2, 2) = (e^4, 9e^4) = e^4(1, 9).$$

b. $f(2, 2) = 4e^4$ will be our point on the tangent plane. The normal vector to the tangent plane is given by

$$\mathbf{n} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1)_{|(2,2)} = (e^4, 9e^4, -1).$$

(This can be memorized or seen from the formula (up to a negative sign) for

$$\mathbf{n} = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}).$$

Therefore, our equation for the tangent plane is

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) &= 0 \\ (e^4, 9e^4, -1) \cdot (x - 2, y - 2, z - 4e^4) &= 0 \\ e^4(x - 2) + 9e^4(y - 2) - z + 4e^4 &= 0 \\ e^4 \left[(x - 2) + 9(y - 2) + 4\right] &= z \end{aligned}$$

c. The linear approximation $L(x, y)$ to the function f at (2,2) is given by the equation of the tangent plane at $(2, 2)$.

$$\begin{aligned} L(x, y) &= e^4 \left[(x - 2) + 9(y - 2) + 4\right] \\ L(1.8, 2.1) &= e^4 \left[(1.8 - 2) + 9(2.1 - 2) + 4\right] \\ &= e^4 \left(-.2 + .9 + 4\right) = 4.7e^4 \end{aligned}$$

7. (15 points) Let $\mathbf{P}(r,\theta) = (r\cos\theta, r\sin\theta)$ be a function $\mathbb{R}^2 \to \mathbb{R}^2$. Notice that $\mathbf{P}$ converts polar coordinates to rectangular coordinates.

a. Calculate $D\mathbf{P}$.

b. Suppose $f(x,y)$ is a function $\mathbb{R}^2 \to \mathbb{R}$, and at the point $\mathbf{x_0} = (0,2)$,

$$\frac{\partial f}{\partial x}(\mathbf{x_0}) = -3, \qquad \frac{\partial f}{\partial y}(\mathbf{x_0}) = 4.$$

Then, what is $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ at this same point? (Hint: consider $D(f \circ \mathbf{P})$, and convert the point $\mathbf{x_0}$ to polar coordinates).

$$DP = \begin{bmatrix} \frac{\partial P_1}{\partial r} & \frac{\partial P_1}{\partial \theta} \\ \frac{\partial P_2}{\partial r} & \frac{\partial P_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Notice that the point $(x,y) = (0,2)$ is equal to $P(2,\pi/2)$. In other words, $P(2,\pi/2) = (0,2)$. This can be seen by converting to polar coordinates. Therefore, we have

$$\begin{aligned} D(f \circ P)_{(r,\theta)} &= Df_{P(r,\theta)} DP_{(r,\theta)} \\ D(f \circ P)_{(2,\pi/2)} &= Df_{(0,2)} DP_{(2,\pi/2)} \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}_{(0,2)} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}_{(2,\pi/2)} \\ &= \begin{bmatrix} -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \frac{\partial f}{\partial r}|_{\mathbf{x_0}} & \frac{\partial f}{\partial \theta}|_{\mathbf{x_0}} \end{bmatrix} = D(f \circ P)_{(2,\pi/2)} = \begin{bmatrix} 4 & 6 \end{bmatrix}$$

Hence, $\frac{\partial f}{\partial r} = 4$, $\frac{\partial f}{\partial \theta} = 6$ at the point $(x,y) = (0,2)$.