Math 205 Final Exam Review Solutions

2.5:19. The chain rule essentially says that $D(f\circ g) = Df\cdot Dg$, where the right hand side indicates matrix multiplication. We use the notation that $x = st, y = tu, z = su$ (obtained from g). Therefore,

$$\begin{aligned}
D(f\circ g) &= Df\cdot Dg = \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 3x^2 & -zeyz & -ye^{yz} \end{bmatrix} \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 3(st)^2 & -suestu^2 & -tue^{stu^2} \end{bmatrix} \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix} \\
&= \begin{bmatrix} t+u & s+u & t+s \\ 3(st)^2t - tu^2e^{stu^2} & 3(st)^2s - su^2e^{stu^2} & -2stue^{stu^2} \end{bmatrix}
\end{aligned}$$

5.3: 16

$$\begin{aligned}
\int_{y=0}^{y=\pi}\int_{x=y}^{x=\pi} \frac{\sin x}{x} dxdy &= \int_{x=0}^{x=\pi}\int_{y=0}^{y=\pi} \frac{\sin x}{x} dydx \\
&= \int_0^{\pi} \sin x dx = 2
\end{aligned}$$

5.4: 12. The region W, when projected to the $x-z$ axis, is the unit circle D, given by $x^2+z^2 = 1$ (with $y = 0$). Over, the region D, the y-values range from 0 to $2-x-z$. Therefore,

$$\begin{aligned}
\iiint_W y\, dV &= \iint_D \int_{y=0}^{y=2-x-z} y\, dy\, dA = \iint_D \frac{1}{2}(2-x-z)^2 dA \\
&= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (2 - r\cos\theta - r\sin\theta)^2 r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 (4r - 4r^3\cos\theta - 4r^2\sin\theta + r^3 + 2r^3\sin\theta\cos\theta)\, dr\, d\theta = \frac{9}{4}
\end{aligned}$$

5.5: 23. Using polar coordinates, we get that

$$\begin{aligned}
\iint_D \cos(x^2+y^2) dA &= \int_0^1 \int_{\pi/3}^{\pi} \cos(r^2) r\, d\theta\, dr = (\pi - \frac{\pi}{3})\int_0^1 \cos(r^2) r\, dr \\
&= \frac{\pi}{3}\sin 1
\end{aligned}$$

5.5: 25. Let D be the disc of radius 2 in the $x-y$ plane.

$$\iiint_w (x^2+y^2+2z^2)\,dV = \iint_D \int_{z=-1}^{z=2} (x^2+y^2+2z^2)\,dz\,dA = \iint_D 3(x^2+y^2)+6\,dA$$
$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (3r^2+6)r\,dr\,d\theta = 48\pi$$

5.6: 5. The temperature function $f(x,y,z)$ is given by

$$f(x,y,z) = k(x^2+y^2+z^2)$$

for some constant k of proportinality. Then,

$$\iiint_W f\,dV = k\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2+z^2)\,dz\,dy\,dx = 8k$$

and the volume of W is 8. Therefore,

$$f_{avg} = \frac{\iiint_W f\,dV}{\iiint_W\,dV} = \frac{8k}{8} = k$$

The set of points of average temperature is then

$$k(x^2+y^2+z^2) = k$$
$$x^2+y^2+z^2 = 1$$

which is just the sphere of radius 1 centered at the origin.

5.8: 10 We use the change of variables

$$u = x+2y, v = y.$$

Then,

$$du\,dv = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dx\,dy = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} dx\,dy$$

After substituting, we have the integral

$$\int_{y=0}^{y=6}\int_{x+2y=0}^{x+2y=1} y^3(x+2y)^2 e^{(x+2y)^3}\,dx\,dy = \int_{y=0}^{y=6} y^3u^2e^{u^3}\,du\,dy$$
$$= \int_0^6 y^3\frac{1}{3}(e-1)\,dy = \frac{6^4}{12}(e-1)$$

6.5: 3. Parameterize the curve by $(x,y) = (a\cos t, a\sin t)$ with $0 \le t \le \pi$.

$$\int_C y\,ds = \int_{t=0}^{t=\pi} y\sqrt{(dx/dt)^2+(dy/dt)^2}dt = \int_0^\pi a\sin t a\,dt = 2a^2$$
$$\int_C ds = \text{Length of } C = \pi a$$
$$[y]_{avg} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

6.5: 21. $\mathbf{x}(t) = (t^3, -t^2, t)$. $\frac{d\mathbf{x}}{dt} = (3t^2, -2t, 1)$.

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} dt = \int_{t=0}^{t=1} (\sin t^3, \cos(-t^2), t^4) \cdot (3t^2, -2t, 1) dt \\ &= \int_0^1 3t^2 \sin t^3 - 2t \cos(-t^2) + t^4 \, dt = -\cos 1 - \sin 1 + \frac{6}{5} \end{aligned}$$

6.5: 22. Let C be the curve traced out as described. Notice that C is traveling in a clockwise direction (as opposed to counterclockwise), so to use Green's theorem properly, we should consider the curve $-C$ going in the opposite direction. Let T be the interior of the triangle formed by the curve C.

$$\begin{aligned} \int_C x^2 y \, dx + (x+y) y \, dy &= -\int_{-C} x^2 y \, dx + (xy + y^2) \, dy = -\iint_T (y - x^2) \, dA \\ &= -\int_{x=0}^{x=1} \int_{y=0}^{y=-x+1} (y - x^2) \, dy \, dx \\ &= -\int_0^1 \frac{1}{2}(-x+1)^2 + x^3 - x^2 \, dx = \frac{11}{12} \end{aligned}$$

6.5: 23. This is easy to do without Green's Theorem.

$$\iint_D dA = \int_{\theta=a}^{\theta=b} \int_{r=0}^{r=f(\theta)} r \, dr \, d\theta = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$$

6.5: 24 Since C is closed (and without any intersection points), we can apply Green's Theorem. Let D be the region bounding C.

$$\int_C f(x) dx + g(y) dy = \iint_D \left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y} \right) = \iint_D 0 \, dA = 0$$