MAT 131 Midterm 1 solutions

1. The U.S. national debt was about \$4 trillion in 1995 and about \$8 trillion in 2005.

   (a) If we model the debt with a linear function of time, what prediction will we have for the debt in 2015?

   Answer After 10 years, the debt has increased by \$4 trillion, so after another 10 years, it will increase by another \$4 trillion. Thus in 2015, the debt would be \$12 trillion.

   (b) If we model the debt with an exponential function of time, what prediction will we have in 2015?

   Answer After 10 years, the debt has doubled. Therefore after another 10 years, the debt will double again, to \$16 trillion.

2. Suppose $f(x) = 3x^2 - x$. Simplify the equation

$$(f \circ f)(x) = x$$

   and find all solutions x.

   (Note: if you get an equation you can't solve, you probably did something wrong.)

   Answer We compute

$$\begin{aligned}(f \circ f)(x) = f\big(f(x)\big) = 3f(x)^2 - f(x) &= 3(3x^2 - x)^2 - (3x^2 - x) \\ &= 27x^4 - 18x^3 + 3x^2 - 3x^2 + x = 27x^4 - 18x^3 + x.\end{aligned}$$

   So the solutions of $(f \circ f)(x) = x$ are the solutions of

$$27x^4 - 18x^3 = 0,$$

   which are $x = 0$ and $x = 2/3$.

3. Suppose the function $f(x)$ has a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 2$. What are the two asymptotes of the function $y = 2f(2x) + 1$? Explain your method.

Answer The graph is being compressed horizontally by a factor of 2; therefore the vertical asymptote at $x = 3$ gets moved to $x = 3/2$. The graph is also being stretched vertically by a factor of 2 and then shifted up by 1 unit, so that the horizontal asymptote $y = 2$ gets moved to $y = 4$ and then shifted to $y = 5$.

4. Consider the function $f(x) = \dfrac{1}{1 - e^{-x}}$.

   (a) Find the inverse function in the form $y = f^{-1}(x)$.

   Answer First we solve for x in terms of y:

$$\begin{aligned} y &= \frac{1}{1 - e^{-x}} \\ 1 - e^{-x} &= \frac{1}{y} \\ e^{-x} &= 1 - \frac{1}{y} \\ e^{x} &= \frac{y}{y-1} \\ x &= \ln\left(\frac{y}{y-1}\right) \end{aligned}$$

   Now we interchange x and y to get

$$f^{-1}(x) = \ln\left(\frac{x}{x-1}\right).$$

   (b) Write down the domain and range of f. Explain your reasoning.

   Answer The domain of f is the set of all x such that $f(x)$ does not involve dividing by 0, which is

$$D = \{x \,|\, x \neq 0\}.$$

   The range of f is the domain of f^{-1}, which we find by the requirement that $y/(y-1)$ must be positive. This implies that either $y > 0$ and $y - 1 > 0$ (so that $y > 1$) or that $y < 0$ and $y - 1 < 0$ (so that $y < 0$). Therefore the range is

$$R = \{y \,|\, y > 1 \text{ or } y < 0\}.$$

5. Compute the limits:

   (a) $\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+5}$

   Answer We try plugging in, since this is a rational function:

$$\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+5} = \frac{1-3+2}{1-4+5} = \frac{0}{2} = 0.$$

   (b) $\lim_{x\to -1} \frac{x^2+9x+8}{x^2-1}$

   Answer Again try plugging in:

$$\lim_{x\to -1} \frac{x^2+9x+8}{x^2-1} = \frac{1-9+8}{1-1} = \frac{0}{0}.$$

   Thus we have to try something else:

$$\lim_{x\to -1} \frac{x^2+9x+8}{x^2-1} = \lim_{x\to -1} \frac{(x+1)(x+8)}{(x+1)(x-1)} = \lim_{x\to -1} \frac{x+8}{x-1} = -\frac{7}{2}.$$

   (c) $\lim_{x\to -1} \frac{x^2-5x+4}{x^2+1}$

   Answer Again plug in:

$$\lim_{x\to -1} \frac{x^2-5x+4}{x^2+1} = \frac{1+5+4}{1+1} = 5.$$

6. (a) How do you know the function $f(x) = x^4 - 17x + 3$ is continuous for all x? (One sentence only.)

   Answer It is a polynomial; all polynomials are continuous.

   (b) Find an interval of length 1 that contains a solution of $f(x) = 0$. What technique are you using?

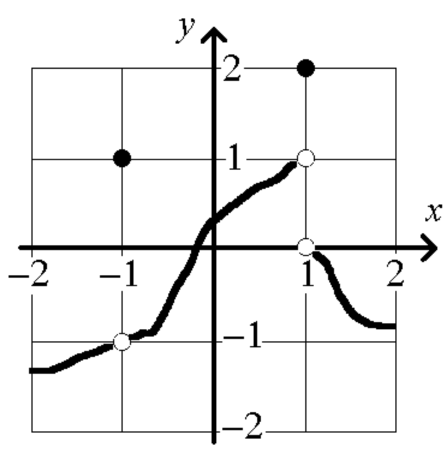
Answer We use the Intermediate Value Theorem. We have to experiment to find an interval by computing special values. $f(0) = 3$, $f(1) = -13$, $f(-1) = 21$, $f(2) = -15$, $f(3) = 30$, $\ldots$

Since f changes sign twice, there are roots between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$. (Either answer is acceptable.)

7. On the given axes, sketch the graph of a function defined at all points of $[-2, 2]$ such that

   - f is continuous at all points of $[-2, 2]$ except for $x = 1$ and $x = -1$.
   - $\lim_{x\to-1} f(x) = -1$
   - $\lim_{x\to1^+} f(x) = 0$
   - $\lim_{x\to1^-} f(x) = 1$
   - $f(1) = 2$

   Answer The only requirement is that $f(-1) \neq -1$, since otherwise f would be continuous at $x = -1$. A possible solution is sketched below.



8. Compute the limits.

   (a) $\lim_{x\to-\infty} \sqrt{x^2 + x - 1} - x$

   Answer As $x \to -\infty$, the term $\sqrt{x^2 + x - 1} \to +\infty$ and also $-x \to +\infty$. Therefore the limit is the sum of two positive infinities and is also positive infinity:

$$\lim_{x\to-\infty} \sqrt{x^2 + x - 1} - x = +\infty + \infty = +\infty.$$

(b) $\lim_{x\to\infty} \sqrt{x^2+x-1} - x$

Answer This limit is of the form $+\infty - \infty$, which is indeterminate. Therefore we have to rationalize the difference of square roots:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x-1} - x &= \lim_{x\to\infty} \frac{\sqrt{x^2+x-1} - x}{1} \frac{\sqrt{x^2+x-1} + x}{\sqrt{x^2+x-1} + x} \\
&= \lim_{x\to\infty} \frac{x^2+x-1-x^2}{\sqrt{x^2+x-1} + x} \\
&= \lim_{x\to\infty} \frac{x-1}{\sqrt{x^2+x-1} + x} \\
&= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{\sqrt{1+\frac{1}{x}-\frac{1}{x^2}} + 1} \\
&= \frac{1-0}{\sqrt{1}+1} \\
&= \frac{1}{2}
\end{aligned}$$