# EXAM

Sample Midterm 1

Math 131

October 9, 2003

# ANSWERS

**Problem 1**. Let $r(x) = \dfrac{2x}{x^2+1}$.

(a) Use the definition of the derivative to compute $r'(a)$.

***Answer***:

$$\begin{aligned}
r'(a) &= \lim_{h\to 0} \frac{r(a+h)-r(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(a+h)}{(a+h)^2+1} - \frac{2a}{a^2+1}}{h} \\
&= \lim_{h\to 0} \frac{2(a+h)(a^2+1) - 2a\left((a+h)^2+1\right)}{h(a^2+1)\left((a+h)^2+1\right)} \\
&= \lim_{h\to 0} \frac{(2a^3+2ha^2+2a+2h) - \left(2a^3+4ha^2+2h^2a+2a\right)}{h(a^2+1)\left((a+h)^2+1\right)} \\
&= \lim_{h\to 0} \frac{-2ha^2+2h-2h^2a}{h(a^2+1)\left((a+h)^2+1\right)} \\
&= \lim_{h\to 0} \frac{-2a^2+2-2ha}{(a^2+1)\left((a+h)^2+1\right)} \\
&= \frac{-2a^2+2}{(a^2+1)^2}.
\end{aligned}$$

(b) There are two points at which the tangent line to the graph of $r$ is horizontal. Give the coordinates of these points.

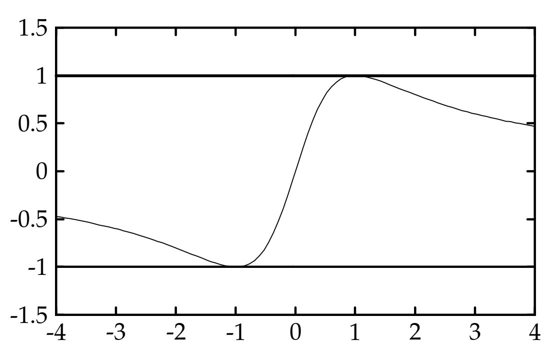
***Answer***:

The tangent line to $y = r(x)$ will be horizontal at $(a, r(a))$ when the slope $r'(a) = \dfrac{-2a^2+2}{(a^2+1)^2}$ is zero. Setting $r'(a) = 0$ gives

$$r'(a) = 0 \Rightarrow \frac{-2a^2+2}{(a^2+1)^2} = 0 \Rightarrow -2a^2+2 = 0 \Rightarrow a = 1 \text{ or } a = -1.$$

So, the tangent line to the graph of $r$ is horizontal at the points $(-1,-1)$ and $(1,1)$.

Here's a little sketch of $y = r(x)$ with its two horizontal tangent lines:



**Problem 2**. Compute:

(a) $\lim_{x\to\infty} \frac{1}{x^2}\sin(x)$

***Answer***:

Because $-1 \le \sin(x) \le 1$ for all $x$, we have $-\frac{1}{x^2} \le \frac{1}{x^2}\sin(x) \le \frac{1}{x^2}$. Since $\lim_{x\to\infty} -\frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^2} = 0$, the squeeze theorem says that $\lim_{x\to\infty} \frac{1}{x^2}\sin(x) = 0$ too.

(b) $\lim_{t\to4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3}$

***Answer***:

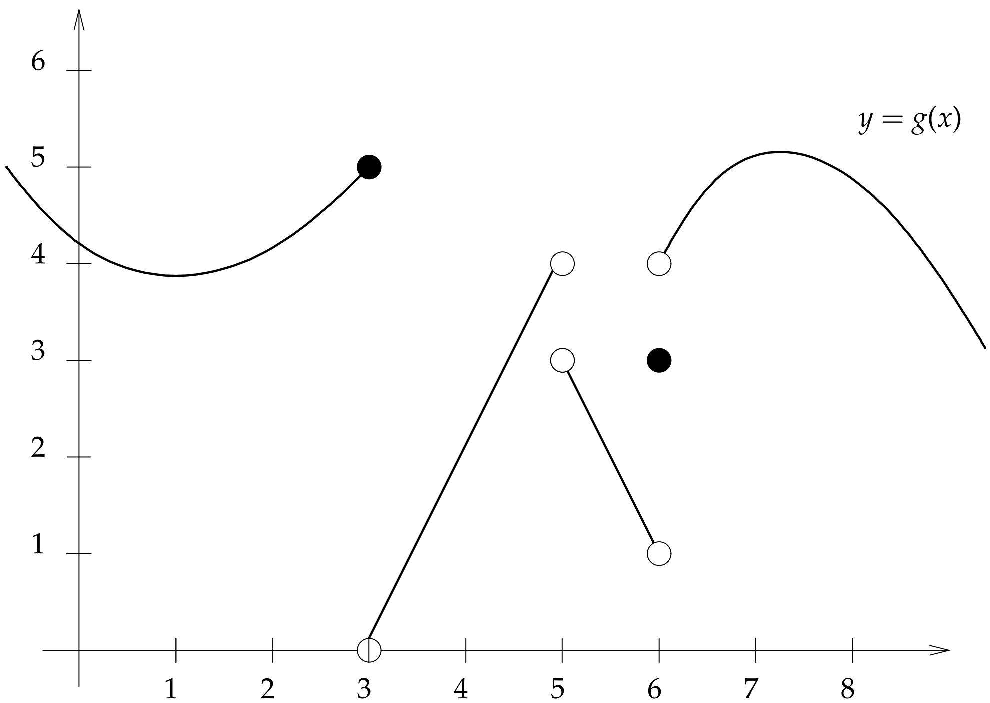
$$\begin{aligned}
\lim_{t\to4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3} &= \lim_{t\to4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3}\left(\frac{\sqrt{2t+1}+3}{\sqrt{2t+1}+3}\right)\left(\frac{\sqrt{t+5}+3}{\sqrt{t+5}+3}\right) \\
&= \lim_{t\to4} \frac{(t+5-9)\left(\sqrt{2t+1}+3\right)}{(2t+1-9)\left(\sqrt{t+5}+3\right)} \\
&= \lim_{t\to4} \frac{(t-4)}{2(t-4)}\left(\frac{\sqrt{2t+1}+3}{\sqrt{t+5}+3}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

(c) $\lim_{x\to2^-} \frac{x^2-3x}{x^2-4}$

***Answer***:

Notice that $\lim_{x\to2^-} x^2-3x = -2$ and $\lim_{x\to2^-} x^2-4 = 0$. When $x$ is near, but less than, 2, $x^2-4 < 0$. So, we find that both the numerator and denominator of $\frac{x^2-3x}{x^2-4}$ are negative for $x$ near but less than 2, with the denominator tending to zero, and the numerator bounded away from zero. We conclude that $\lim_{x\to2^-} \frac{x^2-3x}{x^2-4} = +\infty$.

**Problem 3**.




Use the picture to find:

***Answer***:

(a) $\lim_{x \to 6^+} g(x) = 4.$

(b) $g(6) = 3.$

(c) $\lim_{x \to 6} g(x)$ does not exist.

(d) $g(5)$ does not exist.

(e) $\lim_{x \to 3^-} g(x) = 5.$

(f) $\lim_{x \to 3^+} \dfrac{1}{g(x)} = +\infty.$

(g) $\lim_{x \to 3^+} \dfrac{x-3}{g(x)} = \dfrac{1}{2}$ (note that for $x$ near, but greater than, 3 $g(x)$ looks like $2(x-3)$.)

(h) $\lim_{h \to 0} \dfrac{g(4+h)-2}{h} = 2.$ (here, this is the slope of the tangent line to the graph of $g$ at $(2,2)$.)

(i) $\lim_{x \to 3^-} g(2x) = 1.$ ($= \lim_{x \to 6^-} g(x)$.)

**Problem 4**. A recent study of toxin levels in a stream has yielded the following data:

| $t$ | 1 | 2 | 2.1 | 4 | 6.5 | 7.2 | 8 | 9 | 10 | 10.9 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a(t)$ | 3.12 | 5.4 | 5.7 | 8.15 | 10.10 | 9.63 | 9.21 | 9.31 | 8.98 | 7.82 | 7.62 |

Here, $a(t)$ is the amount of toxin measured at time $t$. Use the table to estimate $a'(2)$ and $a'(11)$.

***Answer***:
Since

$$a'(2) = \lim_{x \to 2} \frac{a(x) - a(2)}{x - 2},$$

it follows that

$$a'(2) \approx \frac{a(x) - a(2)}{x - 2} \text{ if } x \approx 2.$$

So,

$$a'(2) \approx \frac{a(2.1) - a(2)}{2.1 - 2} = \frac{.3}{.1} = 3.$$

Similarly, we approximate

$$a'(11) \approx \frac{a(10.9) - a(11)}{10.9 - 11} = \frac{.2}{-.1} = -2.$$

**Problem 5**. True or False

(a) If $f$ is any function with $f(3) = 5$ and $f(6) = 3$, then there must exist a number $x$ between 3 and 6 with $f(x) = 4$.

***Answer***:

False. For example, look at the function whose graph is sketched in problem 3. There, $g(3) = 5$ and $g(6) = 3$ but there is no $x \in (3, 6)$ with $g(x) = 4$.

(b) The function $f$ defined by

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \leq 1, \\ 5x - 1 & \text{if } x > 1 \end{cases}$$

is continuous at $x = 1$.

***Answer***:

True. We check, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 2x + 1 = 4$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5x - 1 = 4$ and $f(1) = 1$ and find that $\lim_{x\to 1} f(x) = 1 = f(1)$.

(c) $\lim\limits_{r\to 0} \dfrac{1}{r^2} = \infty$.

***Answer***:

True. $\lim\limits_{r\to 0^-} \dfrac{1}{r^2} = \infty$ and $\lim\limits_{r\to 0+} \dfrac{1}{r^2} = \infty$. So, we have the two sided limit $\lim\limits_{r\to 0} \dfrac{1}{r^2} = \infty$.

(d) If $|x - 3| < 0.1$ then $9.9 < 3x + 1 < 10.1$.

***Answer***:

False. Let $x = 2.91$ then $|x - 3| = .09 < 0.1$ but $3x + 1 = 9.73$, which does not lie between 9.9 and 10.1.

(e) If $\lim\limits_{x\to -\infty} g(x) = 14$ then the line $y = 14$ is a horizontal asymptote of the graph of $g$.

***Answer***:

True.

(f) The curve $y = \dfrac{x^3 - 1}{x^2 - 1}$ has a vertical asymptote at $x = 1$.

***Answer***:

False. $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$ which is finite.

(g) The $\lim\limits_{t\to 0} \dfrac{e^t - 1}{t - 1}$ exists and is finite.

***Answer***:

True. This limit is the slope of the tangent line to $y = e^t$ at $(0, 1)$, which by inspection exists and is finite.