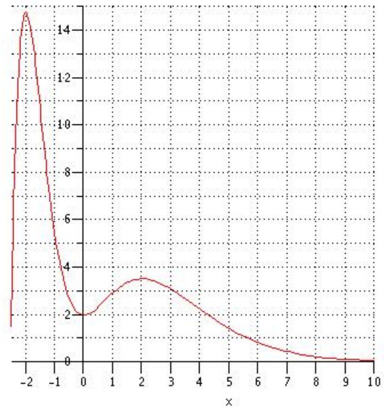
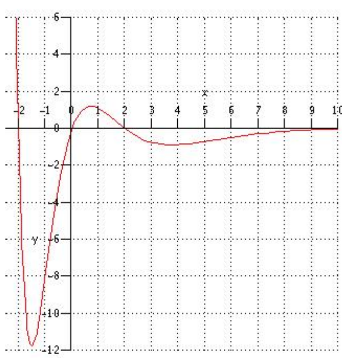
Spring 2007 Midterm 2 Solutions
Part I: Multiple choice questions (5 points each)

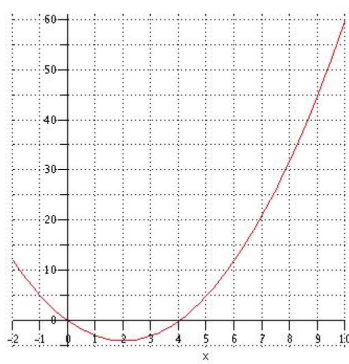


1. Which of the following is the graph of $f'(x)$?
   (a) When $f(x)$ has horizontal tangent lines, $f'(x) = 0$. Notice that this occurs at $x = -2, 0, 2$ and $f'(x)$ should be small but negative for larger x-values.

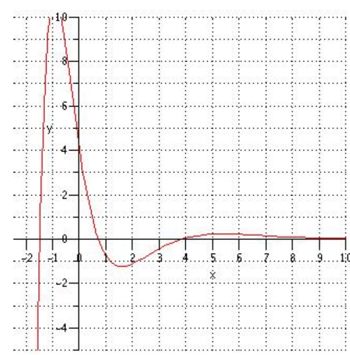
2. Which of the following is the graph of $f''(x)$?
   (c) When $f(x)$ changes concavity, $f''(x) = 0$. This occurs at around $x \approx -1.5, .75, 4$.



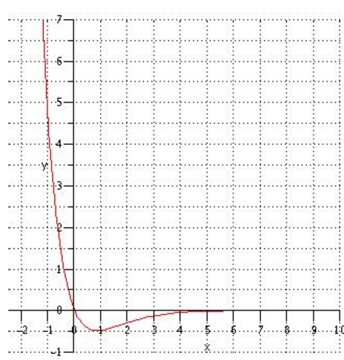
(a)



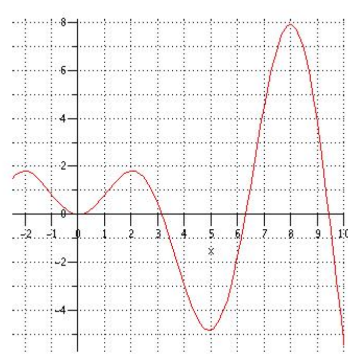
(b)



(c)



(d)



(e)

3. $\lim_{h\to 0} \dfrac{\cos h - 1}{h} =$

(b) 0.
$= f'(0)$ where $f(x) = \cos x$.
$f'(0) = -\sin(0) = 0$.

4. $\dfrac{d}{dx}(e^{2\pi}) =$
(e) 0
$e^{2\pi}$ is a constant.

5. Let $f(x) = (e^x)(x^5 + 4)$. What is $f'(x)$?

(d) $(e^x)(x^5 + 5x^4 + 4)$

$$\begin{aligned} f'(x) &= \left[(e^x)(x^5+4)\right]' = (e^x)'(x^5+4) + (e^x)(x^5+4)' = \\ &= (e^x)(x^5+4) + (e^x)(5x^4) = (e^x)(x^5+5x^4+4). \end{aligned}$$

6. If $y(t) = \sin(\sqrt{3t})$, then $\dfrac{dy}{dt} =$

(b) $\dfrac{3\cos(\sqrt{3t})}{2\sqrt{3t}}$

$$\begin{aligned} \frac{dy}{dt} &= \cos(\sqrt{3t}) \cdot \frac{d}{dt}(\sqrt{3t}) = \cos(\sqrt{3t})\frac{1}{2}(3t)^{-\frac{1}{2}} \cdot \frac{d}{dt}(3t) \\ &= \cos(\sqrt{3t})\frac{1}{2}(3t)^{-\frac{1}{2}}3 = \frac{3\cos(\sqrt{3t})}{2\sqrt{3t}}. \end{aligned}$$

7. Let $h(x) = \ln\left[\dfrac{e^{2x}(x-1)^2}{(x-3)^4}\right]$. Find $h'(x)$. (You may assume $x \neq 1, 3$.)

(c) $2 + \dfrac{2}{x-1} - \dfrac{4}{x-3}$

$$\begin{aligned} h(x) &= 2x + 2\ln(x-1) - 4\ln(x-3) \\ h'(x) &= 2 + 2\frac{(x-1)'}{x-1} - 4\frac{(x-3)'}{x-3} = 2 + \frac{2}{x-1} - \frac{4}{x-3} \end{aligned}$$

8. $W(x) = \cos^4\left(\frac{\sin x}{x^2+1}\right)$. What is $\frac{dW}{dx}$?

There was a mistake. The correct answer does not appear.

$$\begin{aligned}
\frac{dW}{dx} &= \frac{d}{dx}\left[\cos^4\left(\frac{\sin x}{x^2+1}\right)\right] = 4\cos^3\left(\frac{\sin x}{x^2+1}\right)\cdot\frac{d}{dx}\left[\cos(\frac{\sin x}{x^2+1})\right] \\
&= 4\cos^3\left(\frac{\sin x}{x^2+1}\right)\left(-\sin(\frac{\sin x}{x^2+1})\right)\cdot\frac{d}{dx}\left[\frac{\sin x}{x^2+1}\right] \\
&= 4\cos^3\left(\frac{\sin x}{x^2+1}\right)\left(-\sin(\frac{\sin x}{x^2+1})\right)\left[\frac{(x^2+1)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(x^2+1)}{(x^2+1)^4}\right] \\
&= -4\cos^3\left(\frac{\sin x}{x^2+1}\right)\left(\sin(\frac{\sin x}{x^2+1})\right)\left[\frac{(x^2+1)\cos x - (\sin x)(2x)}{(x^2+1)^2}\right]
\end{aligned}$$

9. If $y = x^{2x}$, then $\frac{dy}{dx} =$

(b) $x^{2x}(2\ln x + 2)$

$$\begin{aligned}
y &= x^{2x} \\
\ln y &= \ln(x^{2x}) = 2x\ln x \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(2x\ln x) \\
\frac{1}{y}\frac{dy}{dx} &= 2\ln x + 2x\frac{1}{x} \\
\frac{dy}{dx} &= y(2\ln x + 2) \\
\frac{dy}{dx} &= x^{2x}(2\ln x + 2)
\end{aligned}$$

10. If $h(\theta) = 2^{\cos\theta}$, then $\frac{dh}{d\theta} =$

(a) $-2^{\cos\theta}(\ln 2)(\sin\theta)$

$$\begin{aligned}
\frac{dh}{d\theta} &= \frac{d}{d\theta}(2^{\cos\theta}) \\
&= 2^{\cos\theta}\ln 2\frac{d}{d\theta}(\cos\theta) \\
&= 2^{\cos\theta}\ln 2(-\sin\theta)
\end{aligned}$$

Part II: Partial credit questions (10 points each)
Show all of your work!! Write clearly!

11. Let $g(x) = -\frac{1}{3}x^3 + x^2 + 5$.

(a) For what values of x is $g(x)$ increasing?

$g(x)$ is increasing when $g'(x) > 0$.
$g'(x) = -x^2 + 2x = -x(x-2)$.
$g'(x)$ is a quadratic function and (by several possible means) we can see that

$$-x(x-2) > 0 \text{ when } 0 < x < 2.$$

(b) For what values of x is $g(x)$ concave up?

$g(x)$ is concave up when $g''(x) > 0$.
$g''(x) = -2x + 2$.
$-2x + 2 > 0$ when $x < 1$.

12. Let $f(x) = \sqrt{x}$. Write the equation for the tangent line to $f(x)$ at $x = 25$.

The equation of the tangent line will be

$$y - f(25) = f'(25)(x - 25).$$

$f(25) = \sqrt{25} = 5$.
$f'(x) = \frac{1}{2}(x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
$f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10} = .1$
Therefore, the equation of the tangent line at $x = 25$ is

$$y - 5 = .1(x - 25)$$

or

$$y = .1(x - 25) + 5.$$

(b) Use *linear approximation* to estimate $\sqrt{27}$.

We will use linear approximation for the function $f(x) = \sqrt{x}$ at $x = 25$ since we have an exact decimal value for $\sqrt{25} = 5$. The linearization is exactly the tangent line from part (a). For x close to 25,

$$f(x) \approx L(x) = .1(x - 25) + 5$$
$$\sqrt{27} = f(27) \approx L(27) = .1(27 - 25) + 5 = 5.2$$

13. The equation

$$\frac{x^2}{2} + y^2 = 3$$

gives an ellipse. Find the *slope* of the tangent line to the ellipse at the point $(2, 1)$.

$$\begin{aligned} \frac{d}{dx}\left[\frac{x^2}{2} + y^2\right] &= \frac{d}{dx}(3) \\ x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{2y} \end{aligned}$$

At the point $(2, 1)$,

$$\frac{dy}{dx} = -\frac{2}{2 \cdot 1} = -1.$$

14. Suppose a particle is moving along a circle of radius 1 cm. The angle (in radians) at time t seconds is given by

$$\theta(t) = (t-1)\pi.$$

(a) Calculate $\frac{d\theta}{dt}$ (the angular velocity of the particle) at $t = 1$ second.

$$\frac{d\theta}{dt} = \frac{d}{dt}\left[(t-1)\pi\right] = \pi.$$

Therefore, $\frac{d\theta}{dt} = \pi$ (radian/sec) when $t = 1$.

(b) The (x, y) coordinates of the particle at time t are given by usual trigonometric functions

$$x(\theta) = \cos\theta, \qquad y(\theta) = \sin\theta.$$

Find $\frac{dy}{dt}$ (the vertical velocity of the particle) at $t = 1$ second. Include units.

$$\frac{dy}{dt} = \frac{d}{dt}\left[\sin\theta\right] = \cos\theta \frac{d\theta}{dt}$$

When $t = 1$, $\theta = (1-1)\pi = 0$. Therefore, when $t = 1$ second,

$$\frac{dy}{dt} = (\cos 0)\pi = \pi \text{ cm/sec.}$$

Alternatively, we can write

$$y(\theta(t)) = \sin(t\pi - \pi).$$

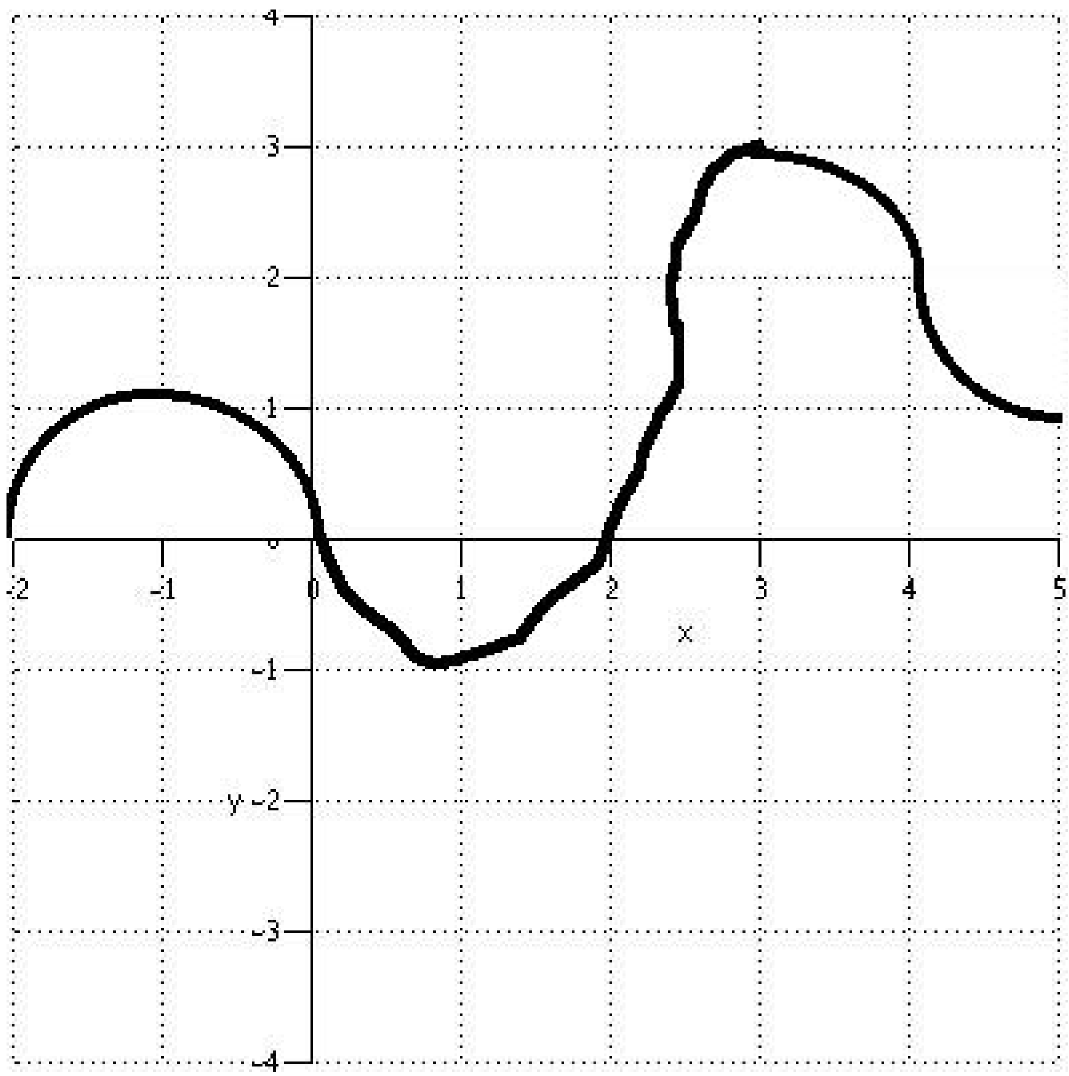
Then, taking the derivative, we see that

$$\frac{dy}{dt} = \cos(t\pi - \pi)\pi$$

and when $t = 1$, $\frac{dy}{dt} = (\cos 0)\pi = \pi$ cm/sec.

15. Draw the graph of a continuous function $h(x)$ that satisfies the following properties:

- $h(3) = 3$
- $h(-2) = h(0) = h(2) = 0$
- $h'(-1) = h'(1) = h'(3) = 0$
- $h''(0) = h''(2.5) = h''(4) = 0$



There are multiple correct graphs one could draw.