Math 131 Name:____________________

Midterm 1 - Solutions ID:____________________

February 20, 2007 Section:____________________

Calculators are not allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 1 hour and 30 minutes to do the test. When you finish, raise your hand and someone will come to collect your exam. You may then leave.

Part I consists of 10 multiple choice questions worth 5 points each. Record your answers by placing an × through one letter for each problem on this answer sheet.

Part II consists of 5 partial credit problems worth a total of 50 points. Write your answer and show **all** your work on the page on which the question appears.

Mark answers in 1-10 with an X.

Good Luck!

1.	a	**X**	c	d	e	6.	**X**	b	c	d	e
2.	a	b	c	**X**	e	7.	a	**X**	c	d	e
3.	**X**	b	c	d	e	8.	a	b	c	d	**X**
4.	a	b	**X**	d	e	9.	a	**X**	c	d	e
5.	a	b	c	**X**	e	10.	**X**	b	c	d	e

For grading use:

1-10	
11	
12	
13	
14	
15	
Total	

Part I: Multiple choice questions (5 points each)

1. Suppose a tank which holds 60 gallons of water is draining from the bottom, and

$$V(t) = 60\left(1 - \frac{t}{6}\right)^2 \qquad 0 \le t \le 6$$

is the volume of water remaining after t minutes. What is the average rate at which water is flowing out over the first 3 minutes?

(a) 10 gal/min

(b) **15 gal/min**

(c) 20 gal/min

(d) 30 gal/min

(e) 45 gal/min

$$\frac{V(3) - V(0)}{3 - 0} = \frac{60(.5)^2 - 60(1)^2}{3} = \frac{15 - 60}{3} = -15 \text{ gal/min.}$$

The volume of water in the tank is decreasing at an average rate of 15 gal/min over the first 3 minutes. The water is flowing out at an average rate of 15 gal/min over the first 3 minutes.

2. Let

$$f(x) = \begin{cases} (cx-2)^2 & x \le 1 \\ x+3 & x > 1 \end{cases}$$

For what value(s) of c is $f(x)$ continuous everywhere?

(a) $c = 0$

(b) $c = \frac{1}{2}$

(c) $c = -2, 2$

(d) c=0,4

(e) No such c exists.

Since both parts of the piecewise function are continous, $f(x)$ is automatically continuous for $x \neq 1$. We therefore only need to impose continuity at $x = 1$.

$$\begin{aligned} f(1) = \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^+} f(x) \\ 4 = \lim_{x\to 1^-} x + 3 &= \lim_{x\to 1^+} (cx-2)^2 \\ 4 = 4 &= (c-2)^2 \\ 4 &= c^2 - 4c + 4 \\ c^2 - 4c &= 0 \\ c(c-4) &= 0 \\ c &= 0, 4 \end{aligned}$$

3. If $g(x) = \sqrt{2}e^{2x+3}$, then the inverse function $g^{-1}(x) =$

(a) $\mathbf{\frac{1}{2}\ln x - \frac{1}{4}\ln 2 - \frac{3}{2}}$

(b) $\frac{1}{2}(\ln x)(\ln \frac{1}{\sqrt{2}}) - \frac{3}{2}$

(c) $\frac{-1}{\sqrt{2}e^{2x+3}}$

(d) $(\ln \sqrt{2})\,(2x+3)$

(e) $\frac{\sqrt{2}}{2}e^{-x} - 3$

$$
\begin{aligned}
y &= \sqrt{2}e^{2x+3} \\
\frac{y}{\sqrt{2}} &= e^{2x+3} \\
\ln(\frac{y}{\sqrt{2}}) &= 2x+3 \\
x &= \frac{\ln(\frac{y}{\sqrt{2}}) - 3}{2} \\
x &= \frac{\ln y - \frac{1}{2}\ln 2 - 3}{2} \\
x &= \frac{1}{2}\ln y - \frac{1}{4}\ln 2 - \frac{3}{2} \\
f^{-1}(x) &= \frac{1}{2}\ln x - \frac{1}{4}\ln 2 - \frac{3}{2}
\end{aligned}
$$

4. $\lim_{x \to 0} x^4 \cos(\frac{1}{x}) =$

   (a) $-\infty$
   (b) -1
   (c) **0**
   (d) 1
   (e) $+\infty$

   For all $x \neq 1$,

   $$-1 \leq \cos(\frac{1}{x}) \leq 1$$

   Therefore, (using the Squeeze Theorem)

   $$\begin{aligned} &-x^4 \leq x^4 \cos(\frac{1}{x}) \leq x^4 \\ &\lim_{x \to 0} -x^4 \leq \lim_{x \to 0} x^4 \cos(\frac{1}{x}) \leq \lim_{x \to 0} x^4 \\ &0 \leq \lim_{x \to 0} x^4 \cos(\frac{1}{x}) \leq 0 \\ &\lim_{x \to 0} x^4 \cos(\frac{1}{x}) = 0 \end{aligned}$$

5. Let

   $$f(x) = \frac{3x^3 - 3x}{2x^3 - 8x}$$

   Which of the following are horizontal assymptotes for $f(x)$?

   (a) $y = -2, 2$
   (b) $y = -1.5, 1.5$
   (c) $y = 0$
   (d) $\mathbf{y = 1.5}$
   (e) No horizontal assymptotes.

   $f(x)$ is a rational function, and the degree of the numberator is equal to the degree of the denominator. Therefore, the horizontal assymptote is given by the ratio of the leading coefficients.

   $$\lim_{x \to \pm\infty} f(x) = \frac{3}{2}.$$

   Alternatively, one can do the following algebraic manipulations:

   $$\lim_{x \to \pm\infty} \frac{3x^3 - 3x}{2x^3 - 8x} = \lim_{x \to \pm\infty} \frac{3x^3 - 3x}{2x^3 - 8x} \left(\frac{x^{-3}}{x^{-3}} \right) = \lim_{x \to \pm\infty} \frac{3 - \frac{3}{x^2}}{2 - \frac{8}{x^2}} = \frac{3}{2}$$

6. $\lim_{x\to 2^-} \frac{3x^3-3x}{2x^3-8x} =$

(a) $-\infty$

(b) -18

(c) 0

(d) $\frac{3}{2}$

(e) $+\infty$

First, notice that if we directly plug in $x = 2$ to $f(x)$, we get $\frac{18}{0}$. Since the denominator is zero, and the numberator is non-zero, this implies that $x = 2$ is a vertical assymptote. We now need to see if $\lim_{x\to 2^-} \frac{3x^3-3x}{2x^3-8x}$ goes to $+\infty$ or $-\infty$. To see this, we need to see if $f(x)$ is positive or negative for x less than, but close to, 2. This can be done by direct substitution, or looking at whether the individual factors are positive or negative. As $x \to 2^-$,

$$\frac{3x^3-3x}{2x^3-8x} = \frac{3x(x-1)(x+1)}{2x(x+2)(x-2)} = \frac{+++}{++-} = \frac{+}{-} = -$$

Therefore, $\lim_{x\to 2^-} \frac{3x^3-3x}{2x^3-8x} = -\infty$.

7. $\lim_{x\to\pi/2} \frac{\cos(x)}{1+\sin(x)} =$

(a) $-\infty$

(b) **0**

(c) $\frac{1}{2}$

(d) $+\infty$

(e) None of the above.

The above function is continuous at all x in the domain. The domain includes $x = \pi/2$. Therefore,

$$\lim_{x\to\pi/2} \frac{\cos(x)}{1+\sin(x)} = \frac{\cos(\pi/2)}{1+\sin(\pi/2)} = \frac{0}{1+1} = 0.$$

8. $\lim_{x \to -\infty} \dfrac{x^5 - 4x^3 + 100}{10x^2 + 45} =$

(a) 0

(b) $\dfrac{1}{10}$

(c) $\dfrac{5}{2}$

(d) $+\infty$

(e) **None of the above.**

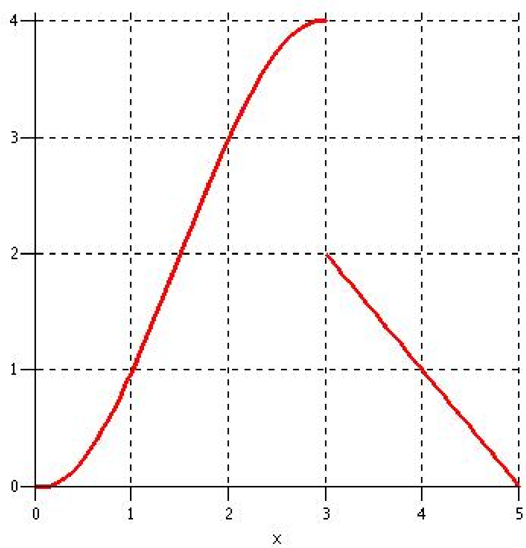
Since the function is a rational function, and the degree of the numerator is greater than the degree of the denominator, then the limit will go off to $+\infty$ or $-\infty$. We determine the sign by looking at the sign of both the numerator and denominator as $x \to -\infty$. The degree of the numerator is odd (it is a degree 5 polynomial), and therefore the numerator is negative as x goes to $-\infty$. Likewise, the denominator is a quadratic function, and hence it is positive as x goes to $-\infty$. Therefore, the quotient is negative as $x \to -\infty$.

Equivalently, one can perform the algebraic manipulation

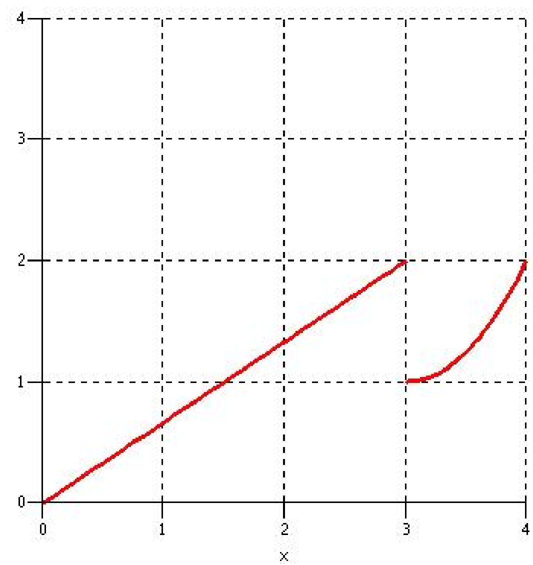
$$\lim_{x \to -\infty} \frac{x^5 - 4x^3 + 100}{10x^2 + 45} = \lim_{x \to -\infty} \frac{x^5 - 4x^3 + 100}{10x^2 + 45} \frac{x^{-2}}{x^{-2}} = \lim_{x \to -\infty} \frac{x^3 - 4x + 100x^{-1}}{10 + 45x^{-2}} = \frac{-\infty}{10} = -\infty.$$

For problems 9-10, use the graphs of $f(x)$ and $g(x)$ below.

$y = f(x)$



$y = g(x)$



9. $\lim_{x \to 3} \dfrac{f(x)}{g(x)} =$

(a) 1

(b) **2**

(c) 4

(d) $+\infty$

(e) Limit does not exist.

We analyze the limits from the left-hand and right-hand sides.

$$\lim_{x \to 3^-} \frac{f(x)}{g(x)} = \frac{4}{2} = 2$$

$$\lim_{x \to 3^+} \frac{f(x)}{g(x)} = \frac{2}{1} = 2.$$

The one-sided limits agree, and

$$\lim_{x \to 3} \frac{f(x)}{g(x)} = 2.$$

10. $\lim_{x \to 4} \frac{f(x) - 1}{x - 4} =$

(a) **-1**

(b) 0

(c) 1

(d) $-\infty$

(e) Cannot be determined

$$\lim_{x \to 4} \frac{f(x) - 1}{x - 4} = f'(4) = -1.$$

The first equality is the definition of the derivative. The value $f'(4)$ is the slope of $f(x)$ at $x = 4$, which is -1 in the graph.

Part II: Partial credit questions (10 points each)

Show all of your work!! Write clearly!

11. (a) State the Intermediate Value Theorem.

IF $f(x)$ is continuous on $[a, b]$, and L is some number in between $f(a)$ and $f(b)$, THEN there exists some c in (a, b) such that $f(c) = L$.

(b) Show that there exists a solution to the equation

$$\cos(x) + x^2 = \pi.$$

Let $f(x) = \cos x + x^2$. Note that $f(x)$ is continuous at all x.

$$f(0) = \cos 0 + 0^2 = 1 < \pi.$$

$$f(\pi) = \cos(2\pi) + (2\pi)^2 = 1 + 4\pi^2 > \pi.$$

Therefore, by the Intermediate Value Theorem, there exists some c in $(0, 2\pi)$ such that

$$f(c) = \cos(c) + c^2 = \pi.$$

(Different x-values can be chosen. The above x-values 0 and 2π were chosen to make $\cos(x)$ easy to evaluate.)

12. Let $f(x) = \sqrt{x+2}$.

(a) Using the definition of the derivative, find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+2+h} - \sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+2+h} - \sqrt{x+2}}{h} \left(\frac{\sqrt{x+2+h} + \sqrt{x+2}}{\sqrt{x+2+h} + \sqrt{x+2}} \right) \\
&= \lim_{h\to 0} \frac{x+2+h-x-2}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \left(\frac{h}{h}\right) \frac{1}{\sqrt{x+2+h} + \sqrt{x+2}} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+2+h} + \sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

(b) Write the equation for the tangent line to the graph of $f(x)$ at $x = 7$.

Slope $m = f'(7) = \dfrac{1}{2\sqrt{7+2}} = \dfrac{1}{6}$.

Point $(x_0, y_0) = (7, f(7)) = (7, \sqrt{7+2}) = (7, 3)$.

$$\begin{aligned}
y - y_0 &= m(x - x_0). \\
y - 3 &= \frac{1}{6}(x - 7)
\end{aligned}$$

13. For which values of x is $w(x)$ continuous?

$$w(x) = \begin{cases} \dfrac{2}{x} & x < 0 \\ \cos(x) + 1 & 0 \leq x \leq 2 \\ \dfrac{1}{(x-1)^2} & x > 2 \end{cases}$$

First, note that each portion of the piecewise function is continuous on its domain. The function $\dfrac{1}{(x-1)^2}$ is only discontinuous at $x = 1$, but $w(x)$ only utilizes $\dfrac{1}{(x-1)^2}$ for $x > 2$. Therefore, we need to check for discontinuities at $x = 0, 2$, which is where the piecewise function switches between functions.

$\lim\limits_{x \to 0^-} w(x) = \lim\limits_{x \to 0^-} \dfrac{2}{x} = -\infty$. Therefore, $w(x)$ is discontinuous at $x = 0$.

$\lim\limits_{x \to 2^-} w(x) = \lim\limits_{x \to 2^-} \cos(x) + 1 = \cos(2) + 1$.

$\lim\limits_{x \to 2^+} w(x) = \lim\limits_{x \to 2^+} \dfrac{1}{(x-1)^2} = \dfrac{1}{(2-1)^2} = 1$.

$\lim\limits_{x \to 2^-} w(x) \neq \lim\limits_{x \to 2^+} w(x)$. Therefore, $w(x)$ is discontinuous at $x = 2$.

The function $w(x)$ is continuous at all x except $x = 0, 2$.

14. Suppose a tall astronaut lands on a fictional planet which happens to have a nice gravitational constant (making calculations simpler). The astronaut throws a rock vertically in the air. The rock's height in meters, t seconds after being thrown, is given by the function

$$s(t) = -5t^2 + 15t + 2.$$

(a) Find the average velocity over the first 2 seconds.

$$\frac{s(2) - s(0)}{2 - 0} = \frac{-5(4) + 15(2) + 2 - 2}{2 - 0} = \frac{10}{2} = 5\text{m/s}.$$

(b) Find the instantaneous velocity at $t = 2$ seconds. (You must use the definition of the derivative).

$$\begin{aligned}
v(2) = s'(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \\
&= \lim_{h \to 0} \frac{-5(2+h)^2 + 15(2+h) + 2 - (-5(2)^2 + 15(2) + 2)}{h} \\
&= \lim_{h \to 0} \frac{-5(4 + 4h + h^2) + 15(2) + 15h + 2 - (-5(4) + 15(2) + 2)}{h} \\
&= \lim_{h \to 0} \frac{-20h - 5h^2 + 15h}{h} \\
&= \lim_{h \to 0} \left(\frac{h}{h}\right)(-5 - 5h) \\
&= \lim_{h \to 0} -5 - 5h \\
&= -5\text{m/s}
\end{aligned}$$

15. Draw the graph of a function $h(x)$ that satisfies the following properties:

- $\lim_{x \to -\infty} h(x) = 1$
- $h(0) = 2$
- $\lim_{x \to 1^-} h(x) = +\infty$
- $\lim_{x \to 1^+} h(x) = -\infty$
- $h(3) = 2$
- $h'(3) = 0$
- $\lim_{x \to +\infty} h(x) = 0$