# Math 639: Lecture 12

## Markov Chains

Bob Hough

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# Markov chains

**Definition**

Given a filtration $\{\mathscr{F}_n\}$, an $\mathscr{F}_n$-adapted stochastic process $\{X_n\}$ taking values in a measurable space $(S, \mathscr{S})$ is called an $\mathscr{F}_n$-Markov chain with state space $(S, \mathscr{S})$ if for any $A \in \mathscr{S}$,

$$\mathrm{Prob}[X_{n+1} \in A | \mathscr{F}_n] = \mathrm{Prob}[X_{n+1} \in A | X_n].$$

Informally, Markov chains are 'memoryless.'

# Markov chains

## Definition

A set function $p : (S, \mathscr{S}) \to [0, 1]$ is a *transition probability* if

1. For each $x \in S$, $A \mapsto p(x, A)$ is a probability measure on $(S, \mathscr{S})$.
2. For each $A \in \mathscr{S}$, $x \mapsto p(x, A)$ is a measurable function.

We say an $\mathscr{F}_n$ Markov chain $\{X_n\}$ has transition probability $p_n(x, A)$ if almost surely

$$\mathrm{Prob}(X_{n+1} \in A | \mathscr{F}_n) = p_n(X_n, A).$$

The Markov chain is called *homogeneous* if $p_n(x, A) = p(x, A)$ for all $n, x \in S$ and $A \in \mathscr{S}$.

# The bounded $\sigma$-algebra

Denote $b\mathscr{S}$ the collection of all *bounded* $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$-valued measurable mappings on $(S, \mathscr{S})$.

# Monotone class theorem

**Theorem**

*Let $\mathscr{A}$ be a $\pi$-system that contains $\Omega$ and let $\mathscr{H}$ be a collection of real-valued functions that satisfies*

1. *If $A \in \mathscr{A}$, then $\mathbf{1}_A \in \mathscr{H}$.*
2. *If $f, g \in \mathscr{H}$, then $f + g$ and $cf \in \mathscr{H}$ for any real $c$.*
3. *If $f_n \in \mathscr{H}$ are non-negative and increase to a bounded function $f$, then $f \in \mathscr{H}$.*

*Then $\mathscr{H}$ contains $b\mathscr{S}$.*

# Markov chains

## Lemma

*If $\{X_n\}$ is an $\mathscr{F}_n$-Markov chain with state space $(S, \mathscr{S})$ and transition probabilities $p_n(\cdot,\cdot)$, then for any $h \in b\mathscr{S}$ and all $k \geqslant 0$,*

$$\mathrm{E}[h(X_{k+1}|\mathscr{F}_k)] = (p_k h)(X_k),$$

*where $h \mapsto (p_k h) : b\mathscr{S} \to b\mathscr{S}$ and $(p_k h)(x) = \int p_k(x, dy) h(y)$ denotes the integral of $h$ under probability measure $p_k(x,\cdot)$.*

# Transition probabilities

**Proposition**

*Given a $\sigma$-finite measure $\nu_1$ on $(X, \mathscr{X})$ and $\nu_2 : X \times \mathscr{S} \mapsto [0, 1]$ such that*

- $B \mapsto \nu_2(x, B)$ *is a probability measure on* $(S, \mathscr{S})$ *for each fixed* $x \in X$
- $x \mapsto \nu_2(x, B)$ *is measurable on* $(X, \mathscr{X})$ *for each fixed* $B \in \mathscr{S}$

*there exists a unique $\sigma$-finite measure $\mu = \nu_1 \otimes \nu_2$ on $(X \times S, \mathscr{X} \times \mathscr{S})$ such that, for all $A \in \mathscr{X}$ and $B \in \mathscr{S}$,*

$$\mu(A \times B) = \int_A \nu_1(dx)\nu_2(x, B).$$

# Markov chains

Given a transition probability $p$ and an *initial distribution* $\mu$ on $(S, \mathscr{S})$, define probability distributions

$$\mathrm{Prob}(X_j \in B_j, 0 \leqslant j \leqslant n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n).$$

**Theorem**

*$X_n$ is a Markov chain with respect to $\mathscr{F}_n = \sigma(X_0, ..., X_n)$ with transition probability $p$.*

# Markov chains

**Theorem**

*If $X_n$ is a Markov chain with transition probabilities $p$ and initial distribution $\mu$, then the finite dimensional distributions are given as above.*

# The law of a Markov chain

## Definition

The law of a Markov chain $\{X_n\}$ with state space $(S, \mathscr{S})$ and initial distribution $\nu$ is the unique probability measure $\mathrm{Prob}_\nu$ on $(S^\infty, \mathscr{S}^\infty)$ (the product space) with finite dimensional distributions

$$\mathrm{Prob}_\nu(\{\mathbf{s} : s_i \in A_i, i = 0, ..., n\}) = \mathrm{Prob}(X_0 \in A_0, ..., X_n \in A_n),$$

for $A_i \in \mathscr{S}$.

# Random walk

## Example (Random walk)

Let $\xi_1, \xi_2, ... \in \mathbb{R}^d$ be i.i.d. with distribution $\mu$. Let $X_0 = x \in \mathbb{R}^d$ and let $X_n = X_0 + \xi_1 + \cdots + \xi_n$. Then $X_n$ is a Markov chain with transition probability

$$p(x, A) = \mu(A - x),$$

where $A - x = \{y - x : y \in A\}$.

# Markov property

The following lemma is useful for proving the Markov property.

**Lemma**

*Let $X$ and $Y$ take values in $(S, \mathscr{S})$. Suppose $\mathscr{F}$ and $Y$ are independent. Let $X \in \mathscr{F}$, $\phi$ be a function with $\mathrm{E}[|\phi(X, Y)|] < \infty$ and let $g(x) = \mathrm{E}[\phi(x, Y)]$. Then*

$$\mathrm{E}[\phi(X, Y)|\mathscr{F}] = g(X).$$

# Examples

To verify that random walk defines a Markov chain, let $\mathscr{F} = \mathscr{F}_n$, $X = X_n$, $Y = \xi_{n+1}$, and $\phi(x, y) = \mathbf{1}(x + y \in A)$. Thus $g(x) = \mu(A - x)$.

**Example**

Let $S = \{0, 1, 2, ...\}$ and

$$p(i,j) = \text{Prob}\left(\sum_{m=1}^{i} \xi_m = j\right)$$

where $\xi_1, \xi_2, ...$ are i.i.d. non-negative integer valued random variables.

# Renewal chain

## Example

Let $S = \{0, 1, 2, ...\}$, $f_k \geqslant 0$, and $\sum_{k=1}^{\infty} f_k = 1$. Set

$$\begin{aligned} p(0, j) &= f_{j+1} & j &\geqslant 0 \\ p(i, i-1) &= 1 & i &\geqslant 1 \\ p(i, j) &= 0 & &\text{otherwise.} \end{aligned}$$

# Ehrenfest chain

**Example**

Let $S = \{0, 1, 2, \ldots, r\}$ and

$$p(k, k+1) = \frac{r-k}{r}$$
$$p(k, k-1) = \frac{k}{r}$$
$$p(i, j) = 0 \text{ otherwise.}$$

This models $r$ particles moving in a split chamber with a small opening connecting the two sides of the chamber. A particle is picked uniformly at random and moved to the other chamber.

# Birth and death chains

**Example**

Let $S = \{0, 1, 2, ...\}$. These chains enforce $p(i, j) = 0$ when $|i - j| > 1$.

# Strong Markov property

### Proposition

*Consider a homogeneous Markov chain $\{X_n\}$ on probability space $(S^\infty, \mathscr{S}^\infty, \mathrm{Prob}_\nu)$. Denote $\theta : S^\infty \to S^\infty$ the shift operator $(\theta\omega)_k = \omega_{k+1}$ and $(\theta^n\omega)_k = \omega_{k+n}$ for $k, n \geqslant 0$. Let $\{h_n\} \subset b\mathscr{S}$ with $\sup_{n,\omega} |h_n(\omega)| < \infty$, and let $\tau$ be a stopping time*

$$\mathrm{E}_\nu[h_\tau(\theta^\tau\omega)|\mathscr{F}_\tau]\mathbf{1}(\tau < \infty) = \mathrm{E}_{X_\tau}[h_\tau]1(\tau < \infty).$$

Here $\mathrm{E}_\nu$ denotes expectation taken with respect to the probability measure $\mathrm{Prob}_\nu$.

# Strong Markov property

## Proof.

- We first check
$$\mathrm{E}_\nu[h(\theta^n\omega)|\mathscr{F}_n] = \mathrm{E}_{X_n}[h],$$
for $h(\omega) = \prod_{\ell=0}^k g_\ell(\omega_\ell)$, with $g_\ell \in b\mathscr{S}$, $\ell = 0, \ldots, k$. The same statement for general $h$ then follows from the monotone class theorem.
- Let $B \in \mathscr{S}^{n+1}$, write $\mu_m = \nu \otimes p \otimes \cdots \otimes p$ and calculate

$$\mathrm{E}_\nu[h(\theta^n\omega)\mathbf{1}_B(\omega_0, \ldots, \omega_n)] = \mu_{n+k}\left[\mathbf{1}_B(x_0, \ldots, x_n)\prod_{\ell=0}^k g_\ell(x_{\ell+n})\right]$$
$$= \mu_n\left[\mathbf{1}_B(x_0, \ldots, x_n)g_0(x_n)\int p(x_n, dy_1)g_1(y_1)\ldots\int p(y_{k-1}, dy_k)g_k(y_k)\right]$$
$$= \mathrm{E}_\nu\left[\mathbf{1}_B(X_0, \ldots, X_n)\,\mathrm{E}_{X_n}[h]\right].$$

□

# Strong Markov property

**Proof.**

- To introduce the stopping time, write $\mathrm{E}_\nu[Y_n|\mathscr{F}_n] = g(n, X_n)$. Conditioning on the value of the stopping time,

$$\mathrm{E}_\nu[h_\tau(\theta^\tau\omega)\mathbf{1}(\tau = k)|\mathscr{F}_\tau] = g(k, X_k)\mathbf{1}(\tau = k) = g(\tau, X_\tau)\mathbf{1}(\tau = k).$$

- Sum in $k$ to complete the estimate.

□

# Invariant measure

## Definition

- A measure $\nu$ on $(S, \mathscr{S})$ such that $\nu(S) > 0$ and $\nu \circ \theta^{-1}(\cdot) = \nu(\cdot)$, i.e. for all $A \in \mathscr{S}^\infty$,
$$\nu(A) = \nu(\{\omega : \theta(\omega) \in A\})$$
is called *shift invariant*. An event $A \in \mathscr{S}^\infty$ is called shift invariant if $A = \theta^{-1}A$.
- We say that a stochastic process $\{X_n\}$ on a state space $\{S, \mathscr{S}\}$ is *stationary* if its joint law $\nu$ is shift invariant.
- A positive measure $\mu$ on a $(S, \mathscr{S})$ is called an *invariant measure* for a transition probability $p(\cdot, \cdot)$ if it defines a shift invariant measure $\text{Prob}_\mu(\cdot)$.

# Invariant measure

**Lemma**

*Suppose a $\sigma$-finite measure $\nu$ and transition probability $p_0(\cdot,\cdot)$ on $(S,\mathscr{S})$ are such that $\nu \otimes p_0(S \times A) = \nu(A)$ for any $A \in \mathscr{S}$. Then, for all $k \geqslant 1$ and $A \in \mathscr{S}^{k+1}$,*

$$\nu \otimes p_0 \otimes \cdots \otimes p_k(S \times A) = \nu \otimes p_1 \otimes \cdots \otimes p_k(A).$$

# Invariant measure

Proof.

- By assumption, $\nu((p_0 f)) = \nu(f)$ for $f = I_A$ and any $A \in \mathscr{S}$. By the monotone class theorem, this extends to all $f \in b\mathscr{S}$.
- Consider $f_k(x) = I_{A_0}(x) p_1 \otimes \cdots \otimes p_k(x, A_1 \times \cdots \times A_k)$. Since $p_j h \in b\mathscr{S}$ for any $h \in b\mathscr{S}$, we have $f_k \in b\mathscr{S}$.
- Observe $\nu(f_k) = \nu \otimes p_1 \otimes \cdots \otimes p_k(A)$ for $A = A_0 \times \cdots \times A_k$ and

$$\begin{aligned} \nu((p_0 f_k)) &= \int_S \nu(dy) \int_{A_0} p_0(y, dx) p_1 \otimes \cdots \otimes p_k(x, A_1 \times \cdots \times A_k) \\ &= \nu \otimes p_0 \otimes \cdots \otimes p_k(S \times A). \end{aligned}$$

The claim for $f_k$ follows by noting $p_1 \otimes \cdots \otimes p_k(x, A_1 \times \cdots \times A_k) \in b\mathscr{S}$, and hence in general by appealing to the monotone class theorem.

□

# Invariant measure

**Theorem**

*A positive $\sigma$-finite measure $\mu(\cdot)$ on $(S, \mathscr{S})$ is an invariant measure for transition probability $p(\cdot, \cdot)$ if and only if $\mu \otimes p(S \times A) = \mu(A)$ for all $A \in \mathscr{S}$.*

# Invariant measure

Proof.

- If $\mu$ is a positive $\sigma$-finite measure, then so are the measures $\mathrm{Prob}_\mu$ and $\mathrm{Prob}_\mu \circ \theta^{-1}$ on $(S^\infty, \mathscr{S}^\infty)$.
- The f.d.d. of $\mathrm{Prob}_\mu$ are the $\sigma$-finite measures $\mu_k(A) = \mu \otimes^k p(A)$ for $A \in \mathscr{S}^{k+1}$.
- By definition of $\theta$, the f.d.d. of $\mathrm{Prob}_\mu \circ \theta^{-1}$ are $\mu_{k+1}(S \times A)$.
- Thus a positive $\sigma$-finite measure $\mu$ is an invariant measure for $p(\cdot,\cdot)$ if and only if $\mu_{k+1}(S \times A) = \mu_k(A)$ for all $k$ and $A \in \mathscr{S}^{k+1}$. This is equivalent to $\mu \otimes p(S \times A) = \mu(A)$.

□

# Countable state spaces

Consider a homogeneous Markov chain $\{X_n\}$ on a countable state space $(S, 2^S)$.

**Theorem**

*For any $x, y \in S$ and non-negative integers $k \leqslant n$,*

$$\mathrm{Prob}_x(X_n = y) = \sum_{z \in S} \mathrm{Prob}_x(X_k = z)\, \mathrm{Prob}_z(X_{n-k} = y).$$

**Proof.**

This follows on conditioning on $X_k$. □

# Harmonic functions

## Definition

We say that $f : S \to \mathbb{R}$ which is bounded above or below, is *super-harmonic* for transition probability $p(x, y)$ at $x \in S$ if

$$f(x) \geqslant \sum_{y \in S} p(x, y) f(y)$$

and *sub-harmonic* at $x$ if

$$f(x) \leqslant \sum_{y \in S} p(x, y) f(y)$$

*harmonic* if equality holds. $f$ is sub/super/harmonic if it is sub/super/harmonic at each $x \in S$.

# Accessible states

**Definition**

- Let $\rho_{xy} = \mathrm{Prob}_x(T_y < \infty)$ be the probability that, started from state $x$, a Markov chain visits state $y$ in finite time.
- If $x \neq y$, and $\rho_{xy} > 0$ then $y$ is *accessible*.
- $x \neq y$ *intercommunicate*, denoted $x \leftrightarrow y$ if each is accessible from the other.
- A non-empty collection of states $C \subset S$ is *irreducible* if each two states of $C$ intercommunicate, and *closed* if no $y \notin C$ is accessible from some $x \in C$.

# Accessible states

**Exercise**

- *Check that if* $\rho_{xy} > 0$ *and* $\rho_{yz} > 0$ *then* $\rho_{xz} > 0$.
- *This implies that intercommunication is an equivalence relation, so the state space splits into maximal irreducible sets, which are connected by a directed graph indicating which class leads to another. This graph is transitive and acyclic.*

# Recurrent states

**Definition**

A state $y \in S$ is called *recurrent*, or *persistent* if $\rho_{yy} = 1$ and *transient* if $\rho_{yy} < 1$.

# Return times

**Theorem**

*Let* $T_y^0 = 0$, *and for* $k \geqslant 1$, $T_y^k = \inf\{n > T_y^{k-1} : X_n = y\}$ *be the time of the* $k$*th return to* $y$. *For any* $x, y \in S$ *and* $k \geqslant 1$,

$$\mathrm{Prob}_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1}.$$

*Let* $N_\infty(y)$ *denote the number of visits to state* $y$. *Then if* $y$ *is transcient,*

$$\mathrm{E}_x[N_\infty(y)] = \frac{\rho_{xy}}{1-\rho_{yy}}.$$

*If* $y$ *is recurrent then* $\mathrm{E}_x[N_\infty(y)]$ *is* $0$ *or* $\infty$ *according as* $\rho_{xy}$ *is* $0$ *or* $1$.

**Proof.**

The formula $\text{Prob}_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1}$ follows from the corresponding formula for iterated stopping times. Calculate

$$\begin{aligned} \text{E}_x[N_\infty(y)] &= \sum_{k=1}^{\infty} \text{Prob}_x(N_\infty(y) \geqslant k) \\ &= \sum_{k=1}^{\infty} \text{Prob}_x(T_y^k < \infty) \\ &= \sum_{k=1}^{\infty} \rho_{xy}\rho_{yy}^{k-1} \end{aligned}$$

which gives the claimed evaluation. □

# Recurrence

**Theorem**

*The following equivalent properties characterize a recurrent state $y$:*

- $\rho_{yy} = 1$
- $\mathrm{Prob}_y(T_y^k < \infty) = 1$ *for all* $k$
- $\mathrm{Prob}_y(X_n = y \text{ } i.o.) = 1$
- $\mathrm{Prob}_y(N_\infty(y) = \infty) = 1$
- $\mathrm{E}_y[N_\infty(y)] = \infty.$

This follows from the previous theorem.

# Decomposition theorem

**Theorem**

*A countable state space $S$ of a homogeneous Markov chain can be partitioned uniquely as*

$$S = T \cup R_1 \cup R_2 \cup \ldots$$

*where $T$ is the set of transient states and the $R_i$ are disjoint, irreducible closed sets of recurrent states with $\rho_{xy} = 1$ whenever $x, y \in R_i$.*

# Decomposition Theorem

**Proof.**

- One easily checks that any pair of interconnected states are either both recurrent or both transient.
- In particular, an irreducible set of states is either transient or recurrent simultaneously. Grouping all transient states together, this provides the decomposition.
- Suppose $x$ is recurrent. If $\rho_{xy} > 0$, then $\rho_{yx} > 0$ since otherwise there is a positive probability of passing from $x$ to $y$ and not returning. This proves the closed condition.

$\square$

# Irreducible, recurrent, transient

**Definition**

A homogeneous Markov chain is *irreducible* if $S$ is irreducible, is *recurrent* if each state is recurrent, and is *transient* if each state is transient.

# Recurrence

### Theorem

*If $F$ is a finite set of transient states then for any initial distribution* $\mathrm{Prob}_\nu(X_n \in F \text{ i.o.}) = 0$. *Any finite closed set $C$ contains at least one recurrent state, and if $C$ is also irreducible then $C$ is recurrent.*

**Proof.**

- If $F$ is a finite set of transient states, then for any $x$, $\mathrm{E}_x[\sum_{f\in F} N_\infty(f)] < \infty$, so the probability that the sites are visited infinitely often is 0.
- If a finite set of sites is closed, then once the chain enters the set, it never leaves. In particular, some site is visited infinitely often and is recurrent.
- If an irreducible set contains a recurrent state, then all states are recurrent, which proves the last claim.

□

# Absorbing states

**Definition**

If a singleton $\{x\}$ is a closed set of a homogeneous Markov chain, then we call $x$ an *absorbing state* for the chain.

# Recurrent states

**Theorem**

*Suppose $S$ is irreducible for a chain $\{X_n\}$ and there exists $h : S \to [0, \infty)$ of finite level sets $G_r = \{x : h(x) < r\}$ that is super-harmonic at $S \backslash G_r$ for some finite $r$. Then $\{X_n\}$ is recurrent.*

# Recurrent states

**Proof.**

- We can assume $S$ is infinite, and that $r_0$ is sufficiently large so that $h$ is superharmonic in $S \backslash G_{r_0}$.
- If $\text{Prob}_x(T_{G_r} < \infty) = 1$ for all $x \in S$, then $S$ contains a recurrent state, hence is recurrent by irreducibility.
- Let $r > r_0$ and let $C_r = G_{r_0} \cup (S \backslash G_r)$. Thus $h$ is super-harmonic in $x \notin C$, so $h(X_{n \wedge \tau_C})$ is a non-negative sup-martingale for $\text{Prob}_x$ for any $x \in S$.
- Since $C^c \subset G_r$ is a finite set, $\text{Prob}_x(\tau_C < \infty) = 1$. Calculate

$$h(x) \geqslant \text{E}_x[h(X_{\tau_C})] \geqslant r \, \text{Prob}_x(\tau_C < \tau_{G_{r_0}}).$$

□

Proof.

- Thus

$$\mathrm{Prob}_x(\tau_{G_{r_0}} < \infty) \geqslant \mathrm{Prob}_x(\tau_{G_{r_0}} < \tau_C) \geqslant 1 - \frac{h(x)}{r}.$$

Letting $r \to \infty$ proves the claim.

□

# Excessive measures

**Definition**

We say that a non-zero $\mu : S \to [0, \infty]$ is an *excessive measure* if

$$\mu(y) \geqslant \sum_{x \in S} \mu(x) p(x, y), \qquad \forall y \in S.$$

# Invariant measure

**Theorem**

*Let $T_z$ denote the possibly infinite return time to a state $z$ by homogeneous Markov chain $\{X_n\}$. Then*

$$\mu_z(y) = \mathrm{E}_z\left[\sum_{n=0}^{T_z-1} \mathbf{1}(X_n = y)\right]$$

*is an excessive measure for $\{X_n\}$, the support of which is the closed set of all states accessible from $z$. If $z$ is a recurrent state then $\mu_z(\cdot)$ is an invariant measure, whose support is closed and recurrent.*

# Invariant measure

**Proof.**

- Set $h_k(\omega, y) = \sum_{n=0}^{T_z(\omega)-1} \mathbf{1}(\omega_{n+k} = y)$. Thus $\mu_z(y) = \mathrm{E}_z[h_0(\omega, y)]$.
- Calculate

$$\begin{aligned}
\mathrm{E}_z[h_1(\omega, y)] &= \mathrm{E}_z\left[\sum_{n=0}^{\infty} \mathbf{1}(T_z > n)\mathbf{1}(X_{n+1} = y) \sum_{x \in S} \mathbf{1}(X_n = x)\right] \\
&= \sum_{x \in S} \sum_{n=0}^{\infty} \mathrm{E}_z\left[\mathbf{1}(T_z > n)\mathbf{1}(X_n = x)\,\mathrm{Prob}_z(X_{n+1} = y | \mathscr{F}_n)\right] \\
&= \sum_{x \in S} \sum_{n=0}^{\infty} \mathrm{E}_z\left[\mathbf{1}(T_z > n)\mathbf{1}(X_n = x)\right] p(x, y) \\
&= \sum_{x \in S} \mu_z(x) p(x, y).
\end{aligned}$$

□

# Invariant measure

**Proof.**

- Observe that if $\omega_0 = z$, $h_0(\omega, y) \geqslant h_1(\omega, y)$ with equality when $y \neq z$ or $T_z(\omega) < \infty$.
- It follows that

$$\mu_z(y) = \mathrm{E}_z[h_0(\omega, y)] \geqslant \mathrm{E}_z[h_1(\omega, y)] = \sum_{x \in S} \mu_z(x) p(x, y),$$

  with equality when $y \neq z$ or $z$ is recurrent. This proves that $\mu_z$ is excessive.
- Iterating, for any $k \geqslant 1$,

$$\mu_z(y) \geqslant \sum_{x \in S} \mu_z(x) \, \mathrm{Prob}_x(X_k = y)$$

  with equality if $z$ is recurrent.

□

Proof.

- If $\rho_{zy} = 0$ then $\mu_z(y) = 0$, while if $\rho_{zy} > 0$ then $\mathrm{Prob}_z(X_k = y) > 0$ for some finite $k$, so that $\mu_z(y) \geqslant \mu_z(z)\,\mathrm{Prob}_z(X_k = y)$. Thus when $z$ is recurrent, the support of $\mu_z$ is its irreducible component.
- If $x \leftrightarrow z$ then $1 = \mu_z(z) \geqslant \mu_z(x)\,\mathrm{Prob}_x(X_k = z)$ for some $k$, whence $\mu_z$ is invariant and $\sigma$-finite.

□

# Invariant measure

**Theorem**

*If $R$ is a recurrent equivalence class of states then the invariant measure whose support is contained in $R$ is unique and has $R$ as its support. In particular, the invariant measure of an irreducible, recurrent chain is unique.*

# Invariant measure

**Proof.**

- Since $R$ is closed, the restriction of $p(\cdot,\cdot)$ to $R$ is a transition probability, so we may assume $S = R$.
- Hence there exists a strictly positive invariant measure $\mu = \mu_z$ on $R$
- Define transition probability $q(x,y) = \frac{\mu(y)p(y,x)}{\mu(x)}$.
- Let $\nu$ be any excessive probability for $p(\cdot,\cdot)$. Then for any $y$,

$$\nu(y) \geqslant \sum_{x\in S} \nu(x)p(x,y) = \sum_{x\in S} \nu(x)q(y,x)\frac{\mu(y)}{\mu(x)}$$

so that $\frac{\nu}{\mu}$ is a superharmonic function for $q$.

□

# Invariant measure

**Proof.**

- By considering paths, we can check that $\rho_{x,y} > 0$ for $p$ implies $\rho_{y,x} > 0$ for $q$, and hence the Markov chain with transition probability $q$ is irreducible.
- Considering loops, the probability of a return from $x$ to $x$ at step $k$ under $p$ is equal to the same probability under $q$ (by running each loop in reverse). Hence the $q$-chain is recurrent.
- Check as an exercise that the only positive super-harmonic function for an irreducible recurrent chain is a constant, and hence $\nu$ is a scalar multiple of $\mu$.

□

# Reversible chains

**Definition**

A non-zero $\mu : S \to [0, \infty)$ is called a *reversible measure* for the transition probability $p(\cdot, \cdot)$ if for all $x, y \in S$, $\mu(x)p(x, y) = \mu(y)p(y, x)$. The transition probability $p(\cdot, \cdot)$ is *reversible* if it has a reversible measure.

# Time-reversed chain

**Definition**

If $\mu(\cdot)$ is an invariant measure for transition probability $p(x, y)$, then $q(x, y) = \mu(y)p(y, x)/\mu(x)$ is a transition probability on the support of $\mu(\cdot)$, call the *adjoint* or *dual* of $p$ with respect to $\mu$. The corresponding Markov chain is called the *time-reversed* chain.

# Random walk on a graph

**Definition**

- A *network* consists of a countable (finite or infinite) set of vertices $V$ with a symmetric weight function $w : V \times V \mapsto [0, \infty)$ (i.e. $w_{xy} = w_{yx}$ for all $x, y \in V$). Set $\mu(x) = \sum_{y \in V} w_{xy}$.
- A *random walk* on the network is a homogeneous Markov chain of state space $V$ and transition probability

$$p(x, y) = \frac{w_{xy}}{\mu(x)}.$$

# Recurrent states

**Definition**

Let $T_z$ denote the first return time to state $z$. A recurrent state $z$ is called *positive recurrent* if $\mathrm{E}_z[T_z] < \infty$ and *null recurrent* otherwise.

# Recurrence

**Theorem**

*If $\pi(\cdot)$ is an invariant probability measure, then all states $z$ with $\pi(z) > 0$ are positive recurrent. Further, if the support of $\pi(\cdot)$ is an irreducible set $R$ of positive recurrent states then $\pi(z) = 1/\operatorname{E}_z[T_z]$ for all $z \in R$.*

**Proof.**

- Starting the chain from the invariant distribution $\pi$ one easily verifies that $\pi$ is supported on recurrent states.
- Calculate, starting from a recurrent state $z$,

$$\mu_z(S) = \sum_{y \in S} \mu_z(y) = \mathrm{E}_x \left[ \sum_{y \in S} \sum_{n=0}^{T_z - 1} \mathbf{1}(X_n = y) \right] = \mathrm{E}_z[T_z].$$

  Thus, if $\mu_z$ is a finite measure then $z$ is positive recurrent.
- If $\pi$ is supported on a single irreducible then $\pi(z) = \frac{\mu_z(z)}{\mu_z(S)} = \frac{1}{\mathrm{E}[T_z]}$.
- To complete the proof, note that an invariant probability measure is a mixture of invariant probability measures supported on single irreducibles.

□

# Markovian coupling

**Theorem**

*Let $\{X_n\}$ and $\{Y_n\}$ be two independent copies of an aperiodic, irreducible Markov chain. Suppose further that the irreducible chain $Z_n = (X_n, Y_n)$ is recurrent. Then, regardless of the initial distribution $(X_0, Y_0)$, the first meeting time $\tau = \min\{\ell \geqslant 0 : X_\ell = Y_\ell\}$ of the two processes is a.s. finite, and for any $n$,*

$$\|\mathscr{L}_{X_n} - \mathscr{L}_{Y_n}\|_{\mathrm{TV}} \leqslant 2\,\mathrm{Prob}(\tau > n).$$

# Markovian coupling

## Proof.

- The Markov chain $Z_n = (X_n, Y_n)$ on $S^2$ is irreducible by independence. Since $\{Z_n\}$ is recurrent, $\tau_z = \min\{\ell \geqslant 0 : Z_\ell = z\}$ is a.s. finite for each $z \in S^2$. Thus,

$$\tau = \inf\{\tau_z : z = (x, x), \text{ some } x \in S\}.$$

- For the remaining claim, let $g \in b\mathscr{S}$ bounded by 1, and verify that, for $k \leqslant n$,

$$\mathbf{1}(\tau = k)\,\mathrm{E}_{X_k}[g(X_{n-k})] = \mathbf{1}(\tau = k)\,\mathrm{E}_{Y_k}[g(Y_{n-k})]$$

or $\mathrm{E}[\mathbf{1}(\tau = k)g(X_n)] = \mathrm{E}[\mathbf{1}(\tau = k)g(Y_n)]$. Thus

$$\mathrm{E}[g(X_n)] - \mathrm{E}[g(Y_n)] = \mathrm{E}[\mathbf{1}(\tau > n)(g(X_n) - g(Y_n))] \leqslant 2\,\mathrm{Prob}(\tau > n).$$

□