MATH 533, SPRING 2022, HW7

DUE MARCH 21

Problem 1. Suppose $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that

$$\lim_{y\to 0} \|y^{-1}(f^{-y} - f) - h\|_p = 0,$$

we call h the *strong L^p derivative* of f and write $h = df/dx$. If $f \in L^p(\mathbb{R}^n)$, L^p derivatives of f are defined similarly. If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and the L^p derivative $\partial_j f$ exists, then prove $\partial_j(f * g)$ exists in the ordinary sense and equals $(\partial_j f) * g$.

Problem 2. Let $\phi \in L^1(\mathbb{R}^n)$ satisfy $|\phi(x)| \le C(1+|x|)^{-n-\epsilon}$ for some $C, \epsilon > 0$, and $\int \phi(x)dx = a$. For $t > 0$, $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$. If $f \in L^p$ define the ϕ-maximal function of f to be $M_\phi f(x) = \sup_{t>0} |f * \phi_t(x)|$. The *Hardy-Littlewood maximal function* Hf is $M_\phi|f|$ where ϕ is the characteristic function of the unit ball, divided by the volume of the ball. Show that there is a constant C, independent of f, such that $M_\phi f \le CHf$.

Problem 3. Young's inequality shows that L^1 is a Banach algebra with convolution as multiplication.

(1) If $\mathcal{I}$ is an ideal in the algebra L^1, prove that its closure is, also.
(2) If $f \in L^1$, the smallest closed ideal in L^1 containing f is the smallest closed subspace of L^1 containing translates of f.

Problem 4. Show that if $f \in L^1(\mathbb{R}^n)$, f is continuous at 0, and $\hat{f} \ge 0$, then $\hat{f} \in L^1$.

Problem 5. Let f be a function on $\mathbb{T}^1$ and $A_r f$ the rth Abel mean of the Fourier series of f. Check that

(1) $A_r f = f * P_r$ where $P_r(x) = \sum_{-\infty}^{\infty} r^{|k|} e^{2\pi i k x}$ is the Poisson kernel for $\mathbb{T}^1$.
(2) $P_r(x) = \frac{1-r^2}{1+r^2-2r\cos 2\pi x}$.

Problem 6. Given $f \in L^1(\mathbb{T}^1)$, let $S_m f(x) = \sum_{-m}^{m} \hat{f}(k) e^{2\pi i k x}$ and

$$\sigma_m f(x) = \sum_{-m}^{m} \hat{f}(k) \left(1 - \frac{|k|}{m+1}\right) e^{2\pi i k x}.$$

Prove the following.

(1) $\sigma_m f = \frac{1}{m+1} \sum_{0}^{m} S_k f$.

(2) If D_k is the kth Dirichlet kernel, we have $\sigma_m f = f * F_m$ where $F_m = \frac{1}{m+1} \sum_{0}^{m} D_k$. F_m is the mth Fejér kernel on $\mathbb{T}^1$.

(3) $F_m(x) = \frac{\sin^2 (m+1)\pi x}{(m+1) \sin^2 \pi x}$.

Problem 7. Prove the following.

(1) If D_m is the mth Dirichlet kernel, $\|D_m\|_1 \to \infty$ as $m \to \infty$.

(2) The Fourier transform is not surjective from $L^1(\mathbb{T}^1)$ to $C_0(\mathbb{Z})$.