MATH 533, SPRING 2022, HW5

DUE IN CLASS, FEBRUARY 28

In the following exercises X is a locally compact Hausdorff space.

Problem 1. If μ is a Radon measure and $f \in L^1(\mu)$, show that $\nu(E) = \int_E f d\mu$ is a Radon measure.

Problem 2. If μ is a Radon measure and $f \in L^1(\mu)$ is real-valued, show that for every $\epsilon > 0$ there are an l.s.c. function g and a u.s.c. function h such that $h \leq f \leq g$ and $\int (g-h) d\mu < \epsilon$.

Problem 3. If μ is a positive Radon measure on X with $\mu(X) = \infty$, show that there exists $f \in C_0(X)$ such that $\int f d\mu = \infty$. Consequently, every positive linear functional on $C_0(X)$ is bounded.

Problem 4. If μ is a σ-finite Radon measure on X and $\nu \in M(X)$, let $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition of ν with respect to μ. Show that ν_1 and ν_2 are Radon.

Problem 5. Show that a sequence $\{f_n\}$ in $C_0(X)$ converges weakly to $f \in C_0(X)$ iff $\sup_n \|f_n\|_u < \infty$ and $f_n \to f$ pointwise.

Problem 6. Find examples of sequences $\{\mu_n\}$ in $M(\mathbb{R})$ such that:

(1) $\mu_n \to 0$ vaguely, but $\|\mu_n\| \not\to 0$.
(2) $\mu_n \to 0$ vaguely, but $\int f d\mu_n \not\to 0$ for some bounded measurable f with compact support.
(3) $\mu_n \geq 0$ and $\mu_n \to \mu$ vaguely, but, there exists $x \in \mathbb{R}$ such that $F_n(x) \not\to F(x)$.

Problem 7. Let μ be a Radon measure on X such that every open set has positive measure. Show that for each $x \in X$ there is a sequence $\{f_n\}$ in $L^1(\mu)$ which converges vaguely in $M(X)$ to the point mass at x.