MATH 533, SPRING 2022, HW3

DUE FEBRUARY 14

Problem 1. Let $\mathcal{Y} = C([0,1])$ and $\mathcal{X} = C^1([0,1])$, both equipped with the uniform norm. Prove:

(1) $\mathcal{X}$ is not complete.

(2) The map $d/dx : \mathcal{X} \to \mathcal{Y}$ is closed but not bounded.

Problem 2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $\{T_n\}$ be a sequence in $L(\mathcal{X}, \mathcal{Y})$ such that $\lim T_n x$ exists for every $x \in \mathcal{X}$. If $Tx = \lim T_n x$, prove that $T \in L(\mathcal{X}, \mathcal{Y})$.

Problem 3. Let $\mathcal{X}$ be a vector space of countably infinite dimension. Prove that there is no norm on $\mathcal{X}$ with respect to which $\mathcal{X}$ is complete.

Problem 4. Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.

(1) Prove that E_n is nowhere dense in $C([0,1])$.

(2) Show the set of nowhere differentiable functions is residual in $C([0,1])$.

Problem 5. Let $\mathcal{X}$ be a normed vector space. Prove the following.

(1) Every weakly convergent sequence in $\mathcal{X}$, and every weak $*$ convergent sequence in $\mathcal{X}^*$, is bounded w.r.t. the norm.

(2) Every weakly compact subset of $\mathcal{X}$, and every weak * compact subset of $\mathcal{X}^*$, is bounded w.r.t. the norm.

(3) If $\mathcal{X}$ is infinite dimensional, every nonempty weakly open set in $\mathcal{X}$, and every nonempty weak * open set in X^* is unbounded w.r.t. the norm.

Problem 6. If $\mathcal{X}$ is a separable normed vector space, show that the weak * topology on the closed unit ball in $\mathcal{X}^*$ is second countable and hence metrizable.

Problem 7. Show that a linear subspace of a normed vector space $\mathcal{X}$ is norm closed iff it is weakly closed.