MATH 533, SPRING 2022 PRACTICE MIDTERM

Each problem is worth 10 points.

Date: March 21, 2022.

Problem 1.

a. State and prove Bessel's inequality for a Hilbert space $\mathcal{H}$.

b. Using Bessel's inequality, or otherwise, prove that if $\mathcal{H}$ has a countable orthonormal basis, then any orthonormal basis of $\mathcal{H}$ is countable.

Problem 2.

a. Let $\mathcal{X}$ be an infinite dimensional normed vector space. Prove that the unit ball $B_1 = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is not compact in the norm topology.

b. Prove Alaoglu's Theorem: Let $\mathcal{X}$ be a Banach space. Prove that the unit ball in $\mathcal{X}^*$

$$B_1 = \{\ell \in \mathcal{X}^* : \|\ell\| \leq 1\}$$

is compact in the weak-* topology. (Hint: identify B_1 with a subspace of $\prod_{x \in \mathcal{X}} [-\|x\|, \|x\|]$.)

Problem 3. Define the following sequence spaces of sequences of real numbers.

- For $p \geq 1$, $\ell_p = \{a = \{a_n\}_{n=1}^{\infty} : \|a\|_p^p = \sum_n |a_n|^p\}$
- $\ell_\infty = \{a = \{a_n\}_{n=1}^{\infty} : \|a\|_\infty = \sup_n |a_n|\}$
- $c_0 = \{a = \{a_n\} : \lim_n a_n = 0, \|a\|_\infty = \sup_n |a_n|\}$.

a. Prove that ℓ_p is separable, but ℓ_∞ is not.

b. Prove $c_0^* = \ell_1$, $\ell_1^* = \ell_\infty$ but $\ell_\infty^* \neq \ell_1$ by using Hahn-Banach. Give an example of a sequence in ℓ_1 which does not converge weakly, but converges weak-*.

Problem 4. Let $\phi \in C_c^\infty(\mathbb{R}^n)$, $\int \phi = 1$, and for real $t > 0$, let $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$. Let $1 \leq p < \infty$ and let $f \in L^p(\mathbb{R}^n)$. Prove that $\phi_t * f \in C^\infty(\mathbb{R}^n)$ and $\phi_t * f \to f$ in L^p as $t \downarrow 0$.

Problem 5. Let μ be a Radon measure on X. Prove that μ is inner regular on Borel sets of finite measure.

Problem 6.

a. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $L(\mathcal{X}, \mathcal{Y})$ be the bounded linear maps between $\mathcal{X}$ and $\mathcal{Y}$. Give a neighborhood base at 0 for the strong and weak operator topologies.

b. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T_n \in L(\mathcal{X}, \mathcal{Y})$ be such that, for each $x \in \mathcal{X}$, $\{T_n x\}$ is Cauchy. Prove that T_n converges strongly to some $T \in L(\mathcal{X}, \mathcal{Y})$.