MATH 322, SPRING 2019 MIDTERM 2, PRACTICE PROBLEMS

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Problem 1. Prove that each two-tensor on $\mathbb{R}^n$ has a unique representation as a sum of a symmetric 2 tensor and an alternating 2 tensor.

Solution. The generic 2 tensor on $\mathbb{R}^n$ may be written

$$\sum_{1\leq i,j\leq n} c_{i,j} x_i \otimes x_j.$$

Write

$$x_i \otimes x_j = \frac{1}{2}(x_i \otimes x_j + x_j \otimes x_i) + \frac{1}{2}(x_i \otimes x_j - x_j \otimes x_i).$$

This expresses $x_i \otimes x_j$ as the sum of a symmetric and alternating tensor. The general claim now holds by linearity.

The representation is unique, since if a 2-tensor h has a representation $h = f_1 + g_1 = f_2 + g_2$ where f_1, f_2 are symmetric and g_1, g_2 are alternating, then $f_1 - f_2 = g_2 - g_1$ is both alternating and symmetric. Acting by a permutation of sign -1, $f_1 - f_2 = -(g_2 - g_1)$, so both are 0.

Problem 2. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^4$ be given by $\alpha(x, y) = \begin{pmatrix} x^3 \\ x^2 y \\ xy^2 \\ y^3 \end{pmatrix}$. Calculate $d\alpha$ and $V(d\alpha)$.

Solution. We have

$$d\alpha(x, y) = \begin{pmatrix} 3x^2 & 0 \\ 2xy & x^2 \\ y^2 & 2xy \\ 0 & 3y^2 \end{pmatrix}$$

and, hence,

$$d\alpha(x,y)^t d\alpha(x,y) = \begin{pmatrix} 9x^4+4x^2y^2+y^4 & 2x^3y+2xy^3 \\ 2x^3y+2xy^3 & x^4+4x^2y^2+9y^4 \end{pmatrix}.$$

It follows that

$$V(d\alpha) = \sqrt{(9x^4+4x^2y^2+y^4)(x^4+4x^2y^2+9y^4)-(2x^3y+2xy^3)^2}.$$

Problem 3. Let $\alpha = x_1 + x_2 + ... + x_n$ and let $\omega = \sum_{j=1}^n (-1)^j x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n$ where the hat indicates that x_j is omitted. Calculate $\alpha \wedge \omega$.

Solution. Recall that, in a tensor $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$, if any index is repeated, the wedge product is 0. Hence, expanding the two sums,

$$\alpha \wedge \omega = \sum_{j=1}^n (-1)^j x_j \wedge x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.$$

Moving x_j into the missing slot requires $j-1$ transpositions, so

$$(-1)^j x_j \wedge x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n = -x_1 \wedge x_2 \wedge \cdots \wedge x_n$$

and, hence, $\alpha \wedge \omega = -n x_1 \wedge \cdots \wedge x_n$.

Problem 4. On midterm 1, a coordinate patch α was defined from an open neighborhood U of 0 in $\mathbb{R}^{\frac{n(n-1)}{2}}$ to a neighborhood of the identity in the orthogonal group $O_n = \{M \in \mathbb{R}^{n\times n},\ M^t M = I\}$. Let this coordinate patch be $\alpha : U \to V$. Let O be any orthogonal matrix, and define $\alpha_O : U \to O \cdot V = \{O \cdot M : M \in V\}$ by $\alpha_O(\underline{x}) = O \cdot \alpha(\underline{x})$.

a. Prove that $V(d\alpha) = V(d\alpha_O)$. Deduce that the volume of V and $O \cdot V$ are equal.
b. Show that the same is true for $\alpha^O(\underline{x}) = \alpha(\underline{x}) \cdot O$.

This says that the volume form $V(d\beta)$ on the orthogonal group is left and right translation invariant, and hence is a scalar multiple of Haar measure.

Solution. Recall that, since O is orthogonal, its rows are orthogonal, so

$$\sum_{m=1}^n O_{im} O_{jm} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and its columns are orthogonal, so

$$\sum_{m=1}^{n} O_{mi}O_{mj} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} .$$

a. Let $\alpha(\underline{x}) = \alpha_{i,j}(\underline{x})$ with $1 \leq i,j \leq n$ be an $n\times n$ orthogonal matrix. Let $N = \frac{n(n-1)}{2}$. The matrix

$$M_\alpha(\underline{x}) = (d\alpha(\underline{x}))^t d\alpha(\underline{x})$$

is an $N \times N$ matrix with entries

$$M_\alpha(\underline{x})_{k,\ell} = \sum_{i,j=1}^{n} \frac{\partial \alpha(\underline{x})_{i,j}}{\partial x_k}\frac{\partial \alpha(\underline{x})_{i,j}}{\partial x_\ell},$$

and $V(d\alpha(\underline{x})) = \det(M_\alpha(\underline{x}))^{\frac{1}{2}}$. By matrix multiplication,

$$\alpha_O(\underline{x})_{i,j} = \sum_{r=1}^{n} O_{i,r}\alpha(\underline{x})_{r,j}$$

and, hence,

$$\frac{\partial \alpha_O(\underline{x})_{i,j}}{\partial x_k} = \sum_{r=1}^{n} O_{i,r}\frac{\partial \alpha(\underline{x})_{r,j}}{\partial x_k}.$$

It follows that

$$\begin{aligned} M_{\alpha_O}(\underline{x})_{k,\ell} &= \sum_{i,j=1}^{n} \frac{\partial \alpha_O(\underline{x})_{i,j}}{\partial x_k}\frac{\partial \alpha_O(\underline{x})_{i,j}}{\partial x_\ell} \\ &= \sum_{i,j=1}^{n}\sum_{r_1=1}^{n}\sum_{r_2=1}^{n} O_{i,r_1}O_{i,r_2}\frac{\partial \alpha(\underline{x})_{r_1,j}}{\partial x_k}\frac{\partial \alpha(\underline{x})_{r_2,j}}{\partial x_\ell}. \end{aligned}$$

Summing over i selects $r_1 = r_2 = r$, say, so that

$$M_{\alpha_O}(\underline{x})_{k,\ell} = \sum_{r,j=1}^{n} \frac{\partial \alpha(\underline{x})_{r,j}}{\partial x_k}\frac{\partial \alpha(\underline{x})_{r,j}}{\partial x_\ell} = M_\alpha(\underline{x})_{k,\ell}.$$

Hence $V(d\alpha_O(\underline{x})) = V(d\alpha(\underline{x}))$ since the M matrices are equal. Similarly,

$$\frac{\partial \alpha^O(\underline{x})_{i,j}}{\partial x_k} = \sum_{r=1}^n \frac{\partial \alpha(\underline{x})_{i,r}}{\partial x_k} O_{r,j}$$

and the argument works as before, although the summation over j is now used to select $r_1 = r_2$.

Problem 5. Let $S^{n-1} = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\|_2 = 1\}$ be the unit sphere in $\mathbb{R}^n$. Parameterize the part of the unit sphere in the first octant by $\alpha : U \to V$, where $U = \{\underline{x} \in \mathbb{R}^{n-1}_{>0}, \|\underline{x}\|_2 < 1\}$ and

$$\alpha(\underline{x}) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} \end{pmatrix}.$$

Let $A : U \times \mathbb{R}_{>0} \to \mathbb{R}^n_{>0}$ be defined by

$$A(\underline{x}, t) = \begin{pmatrix} tx_1 \\ tx_2 \\ \vdots \\ tx_{n-1} \\ t\sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} \end{pmatrix}.$$

a. Show that

$$V(d\alpha) = \frac{1}{\sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}}$$

and $V(dA) = t^{n-1} V(d\alpha)$.

b. Show that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

by squaring and switching to polar coordinates. Then calculate

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-\frac{x^2}{2}} dx.$$

Using this or otherwise, calculate the moments of the Gaussian distribution,

$$M_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-\frac{x^2}{2}} dx,$$

$M_k = 0$ if k is odd and $M_{2k} = \frac{(2k)!}{2^k k!}$.

c. Using this or otherwise, calculate the moments of the coordinates of the sphere S^{n-1},

$$m_k = \frac{\int_{S^{k-1}} x_1^k dV}{V(S^{k-1})}$$

by first doing the same calculation for

$$M_k = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} x_1^k e^{-\frac{x_1^2+\cdots+x_n^2}{2}} d\underline{x}$$

and switching to the parameterization of $\mathbb{R}^n_{>0}$ given in part a. You may express your answers in terms of the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx.$$

Solution.

a. Calculate

$$d\alpha(\underline{x}) = \begin{pmatrix} I_{n-1} \\ -\frac{\underline{x}^t}{\sqrt{1-\|\underline{x}\|_2^2}} \end{pmatrix}.$$

From the formula, $V(X)^2 = \sum_I \det(X_I)^2$ it follows that

$$V(d\alpha(\underline{x}))^2 = 1 + \sum_{j=1}^{n-1} \left(\det \begin{pmatrix} I_{n-1}^j \\ \frac{-\underline{x}^t}{\sqrt{1-\|\underline{x}\|_2^2}} \end{pmatrix} \right)^2$$

where I_{n-1}^j indicates the $(n-1)\times(n-1)$ identity matrix with jth row deleted. This proves

$$V(d\alpha(\underline{x})) = \sqrt{1 + \frac{\|\underline{x}\|_2^2}{1-\|\underline{x}\|_2^2}} = \frac{1}{\sqrt{1-\|\underline{x}\|_2^2}}.$$

Next, calculate

$$dA(\underline{x}, t) = \begin{pmatrix} t \cdot I_{n-1} & \underline{x} \\ \frac{-t\underline{x}^t}{\sqrt{1-\|\underline{x}\|_2^2}} & \sqrt{1-\|\underline{x}\|_2^2} \end{pmatrix}.$$

This is an $n \times n$ matrix, so $V(dA(\underline{x}, t)) = |\det(dA(\underline{x}, t))|$. Since the first n columns are scaled by t, and pulling out a factor of $\frac{1}{\sqrt{1-\|\underline{x}\|_2^2}}$ from the bottom row,

$$\begin{aligned} V(dA(\underline{x}, t)) &= \frac{t^{n-1}}{\sqrt{1-\|\underline{x}\|_2^2}} \left|\det \begin{pmatrix} I_{n-1} & \underline{x} \\ -\underline{x}^t & 1-\|\underline{x}\|_2^2 \end{pmatrix}\right| \\ &= \frac{t^{n-1}}{\sqrt{1-\|\underline{x}\|_2^2}} \left|\det \begin{pmatrix} I_{n-1} & 0 \\ -\underline{x}^t & 1 \end{pmatrix}\right| \\ &= \frac{t^{n-1}}{\sqrt{1-\|\underline{x}\|_2^2}}, \end{aligned}$$

where the next to last line follows from performing column operations in the determinant.

b. We have

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)^2 &= \int_{\mathbb{R}^2} e^{\frac{-x^2-y^2}{2}} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta = 2\pi. \end{aligned}$$

Similarly,

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt - \frac{x^2}{2}} dx \\ &= e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx = e^{\frac{t^2}{2}}. \end{aligned}$$

Expanding $e^{xt} = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!}$, and exchanging the summation and integration, which is justified by absolute convergence,

$$\begin{aligned} e^{\frac{t^2}{2}} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (xt)^n e^{-\frac{x^2}{2}} dx \\ &= \sum_{n=0}^{\infty} \frac{M_{2n} t^{2n}}{(2n)!}, \end{aligned}$$

since the odd n terms integrate to 0 by symmetry. Since the power series have infinite radius of convergence, equating coefficients,

$$M_{2n} = \frac{(2n)!}{2^n n!}.$$

c. We have

$$M_k = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} x_1^k e^{-\frac{\|\underline{x}\|_2^2}{2}} d\underline{x}$$

since integration in each x_j, $j > 1$ integrates to 1. Using the parametrization of part a.,

$$\begin{aligned} M_k &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\underline{x} \in S^{n-1}} \int_{t=0}^{\infty} (tx_1)^k e^{-\frac{t^2}{2}} t^{n-1} dt dV(\underline{x}) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\underline{x} \in S^{n-1}} x_1^k dV(\underline{x}) \int_0^{\infty} t^{n+k} e^{-\frac{t^2}{2}} \frac{dt}{t}. \end{aligned}$$

Substituting $u = t^2$, so $\frac{1}{2} \log u = \log t$ and, hence, $\frac{1}{2} \frac{du}{u} = \frac{dt}{t}$, the second integral is

$$\frac{1}{2} \int_0^{\infty} u^{\frac{n+k}{2}} e^{-\frac{u}{2}} \frac{du}{u} = 2^{\frac{n+k}{2} - 1} \Gamma\left(\frac{n+k}{2}\right).$$

Thus

$$M_{2k} = \frac{(2k)!}{2^k k!} = 2^{\frac{n+2k}{2}-1}\Gamma\left(\frac{n+2k}{2}\right)\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\underline{x}\in S^{n-1}} x_1^{2k} dV(\underline{x}),$$

or

$$\begin{aligned} m_{2k} &= \frac{\int_{S^{n-1}} x_1^{2k} dV(\underline{x})}{\int_{S^{n-1}} dV(\underline{x})} \\ &= \frac{(2k)!}{2^{2k}k!}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+k\right)} \\ &= \frac{(2k)!}{2^{2k}k!}\frac{1}{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)}. \end{aligned}$$

The odd moments vanish by symmetry.

Problem 6. Let $e_1, e_2, ..., e_n$ be the standard basis vectors in $\mathbb{R}^n$, and let $x_1, ..., x_n$ be the dual basis. Given an elementary alternating k form,

$$\alpha = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$$

define the dual elementary alternating $n-k$ form by letting $j_1, ..., j_{n-k}$ be the complementary indices in $\{1, 2, ..., n\}$, so $\{i_1, ..., i_k\} \cup \{j_1, ..., j_{n-k}\} = \{1, 2, ..., n\}$ and defining

$$\alpha^* = \epsilon x_{j_1} \wedge \cdots \wedge x_{j_{n-k}}$$

where the sign ϵ is chosen so that $\alpha \wedge \alpha^* = x_1 \wedge \cdots \wedge x_n$. Note that $x_1 \wedge \cdots \wedge x_n$ has dual form 1. Extend duality linearly, so if

$$\beta = \sum_I b_I x_{i_1} \wedge \cdots \wedge x_{i_k}$$

then

$$\beta^* = \sum_I b_I (x_{i_1} \wedge \cdots \wedge x_{i_k})^*.$$

a. Prove that

$$\langle \alpha, \beta \rangle = (\alpha \wedge \beta^*)^*$$

defines an inner product on $\mathcal{A}^k(\mathbb{R}^n)$ which makes the elementary k forms an orthonormal basis.

b. Given an $n \times k$ matrix M with column vectors $v_1, ..., v_k \in \mathbb{R}^n$, with entries $v_i = \sum_{j=1}^n m_{ji} e_j$, let the dual form be $\ell_i = \sum_{j=1}^n m_{ji} x_j$. Let

$$\omega = \ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_k.$$

Prove that

$$V(M)^2 = \langle \omega, \omega \rangle = (\omega \wedge \omega^*)^*,$$

where $V(M)$ denotes the k-dimensional volume of the parallelpiped spanned by the column vectors of M.

Solution.

a. Note that $\alpha \wedge \beta^*$ is an alternating n form, hence is a multiple of $x_1 \wedge \cdots \wedge x_n$, so $(\alpha \wedge \beta^*)^*$ is a scalar. Since both the wedge product and dual are linear, $(\alpha \wedge \beta^*)^*$ is a bilinear form. Let

$$\alpha = \sum c_I x_{i_1} \wedge \cdots \wedge x_{i_k}.$$

If $I \neq J$ then $x_{i_1} \wedge \cdots \wedge x_{i_k}$ and $(x_{j_1} \wedge \cdots \wedge x_{j_k})^*$ have some index in common, so that

$$(x_{i_1} \wedge \cdots x_{i_k}) \wedge (x_{j_1} \wedge \cdots \wedge x_{j_k})^* = 0,$$

and, hence,

$$\alpha \wedge \alpha^* = \sum_{I,J} c_I c_J (x_{i_1} \wedge \cdots x_{i_k}) \wedge (x_{j_1} \wedge \cdots \wedge x_{j_k})^* = \sum_I c_I^2 x_1 \wedge \cdots \wedge x_n$$

so $\langle \alpha, \alpha \rangle = \sum_I c_I^2$. This proves that $\langle , \rangle$ is non-degenerate, and that the elementary alternating forms are an orthonormal basis for this inner product.

b. We expand

$$\begin{aligned} \omega = \ell_1 \wedge \cdots \wedge \ell_k &= \sum_{i_1=1}^n m_{i_1,1} x_{i_1} \wedge \cdots \wedge \sum_{i_k=1}^n m_{i_k,k} x_{i_k} \\ &= \sum_{i_1,\ldots,i_k=1}^n m_{i_1,1} \cdots m_{i_k,k} x_{i_1} \wedge \cdots \wedge x_{i_k}. \end{aligned}$$

Applying a permutation so that each set of indices is in order, and collecting the product according to the elementary alternating k form, this is

$$\begin{aligned}\omega &= \sum_I \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) m_{i_{\sigma(1)},1} \cdots m_{i_{\sigma(k)},k} x_{i_1} \wedge \cdots \wedge x_{i_k} \\ &= \sum_I (\det M_I) x_{i_1} \wedge \cdots \wedge x_{i_k}\end{aligned}$$

and, hence,

$$\|\omega\|^2 = \sum_I (\det M_I)^2 = V(M)^2$$

by the Pythagorean theorem.

Problem 7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map, which acts on the standard basis $e_1, ..., e_n$ by

$$Te_i = \sum_{j=1}^n m_{ij} e_j.$$

Let $x_1, ..., x_n$ be the dual basis to $e_1, ..., e_n$. Prove

$$T^*(x_1 \wedge \cdots \wedge x_n) = \det(m_{ij}) x_1 \wedge \cdots \wedge x_n.$$

Solution. Since $T^*(x_1 \wedge \cdots \wedge x_n)$ is an alternating n form, $T^*(x_1 \wedge \cdots \wedge x_n) = c(x_1 \wedge \cdots \wedge x_n)$. Hence

$$\begin{aligned}c &= T^*(x_1 \wedge \cdots \wedge x_n)(e_1, ..., e_n) \\ &= x_1 \wedge \cdots \wedge x_n(Te_1, ..., Te_n) \\ &= x_1 \wedge \cdots \wedge x_n \left(\sum_{i_1=1}^n m_{1i_1} e_{i_1}, \sum_{i_2=1}^n m_{2i_2} e_{i_2}, \cdots, \sum_{i_n=1}^n m_{ni_n} e_{i_n} \right) \\ &= \sum_{i_1,...,i_n=1}^n m_{1i_1} \cdots m_{ni_n} x_1 \wedge \cdots \wedge x_n(e_{i_1}, ..., e_{i_n}).\end{aligned}$$

Those tuples with a repeated index among $i_1, ..., i_n$ evaluate to 0, and the remaining ones are a permutation of $1, 2, ..., n$, with evaluation on $x_1 \wedge \cdots \wedge x_n$

equal to the sign of the permutation. Hence

$$c = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) m_{1\sigma(1)} \cdots m_{n\sigma(n)} = \det(m_{ij}).$$