MATH 322, SPRING 2019 MIDTERM 2, PRACTICE PROBLEMS

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Problem 1. Prove that each two-tensor on $\mathbb{R}^n$ has a unique representation as a sum of a symmetric 2 tensor and an alternating 2 tensor.

Problem 2. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^4$ be given by $\alpha(x, y) = \begin{pmatrix} x^3 \\ x^2 y \\ xy^2 \\ y^3 \end{pmatrix}$. Calculate $d\alpha$ and $V(d\alpha)$.

Problem 3. Let $\alpha = x_1 + x_2 + ... + x_n$ and let $\omega = \sum_{j=1}^n (-1)^j x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n$ where the hat indicates that x_j is omitted. Express $\alpha \wedge \omega$ in elementary alternating tensors.

Problem 4. On midterm 1, a coordinate patch α was defined from an open neighborhood U of 0 in $\mathbb{R}^{\frac{n(n-1)}{2}}$ to a neighborhood of the identity in the orthogonal group $O_n = \{M \in \mathbb{R}^{n\times n},\ M^t M = I\}$. Let this coordinate patch be $\alpha : U \to V$. Let O be any orthogonal matrix, and define $\alpha_O : U \to O \cdot V = \{O \cdot M : M \in V\}$ by $\alpha_O(\underline{x}) = O \cdot \alpha(\underline{x})$.

a. Prove that $V(d\alpha) = V(d\alpha_O)$. Deduce that the volume of V and $O \cdot V$ are equal.
b. Show that the same is true for $\alpha^O(\underline{x}) = \alpha(\underline{x}) \cdot O$.

This says that the volume form $V(d\beta)$ on the orthogonal group is left and right translation invariant, and hence is a scalar multiple of Haar measure.

Problem 5. Let $S^{n-1} = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\|_2 = 1\}$ be the unit sphere in $\mathbb{R}^n$. Parameterize the part of the unit sphere in the first octant by $\alpha : U \to V$, where $U = \{\underline{x} \in \mathbb{R}^{n-1}_{>0}, \|\underline{x}\|_2 < 1\}$ and

$$\alpha(\underline{x}) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} \end{pmatrix}.$$

Let $A : U \times \mathbb{R}_{>0} \to \mathbb{R}^n_{>0}$ be defined by

$$A(\underline{x}, t) = \begin{pmatrix} tx_1 \\ tx_2 \\ \vdots \\ tx_{n-1} \\ t\sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} \end{pmatrix}.$$

a. Show that
$$V(d\alpha) = \frac{1}{\sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}}$$
and $V(dA) = t^{n-1}V(d\alpha)$.

b. Show that
$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$
by squaring and switching to polar coordinates. Then calculate
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-\frac{x^2}{2}} dx.$$
Using this or otherwise, calculate the moments of the Gaussian distribution,
$$M_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-\frac{x^2}{2}} dx,$$
$M_k = 0$ if k is odd and $M_{2k} = \frac{(2k)!}{2^k k!}$.

c. Using this or otherwise, calculate the moments of the coordinates of the sphere S^{n-1},

$$m_k = \frac{\int_{S^{k-1}} x_1^k dV}{V(S^{k-1})}$$

by first doing the same calculation for

$$M_k = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} x_1^k e^{-\frac{x_1^2+\cdots+x_n^2}{2}} d\underline{x}$$

and switching to the parameterization of $\mathbb{R}^n_{>0}$ given in part a. You may express your answers in terms of the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

Problem 6. Let $e_1, e_2, ..., e_n$ be the standard basis vectors in $\mathbb{R}^n$, and let $x_1, ..., x_n$ be the dual basis. Given an elementary alternating k form,

$$\alpha = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$$

define the dual elementary alternating $n-k$ form by letting $j_1, ..., j_{n-k}$ be the complementary indices in $\{1, 2, ..., n\}$, so $\{i_1, ..., i_k\} \cup \{j_1, ..., j_{n-k}\} = \{1, 2, ..., n\}$ and defining

$$\alpha^* = \epsilon x_{j_1} \wedge \cdots \wedge x_{j_{n-k}}$$

where the sign ϵ is chosen so that $\alpha \wedge \alpha^* = x_1 \wedge \cdots \wedge x_n$. Note that $x_1 \wedge \cdots \wedge x_n$ has dual form 1. Extend duality linearly, so if

$$\beta = \sum_I b_I x_{i_1} \wedge \cdots \wedge x_{i_k}$$

then

$$\beta^* = \sum_I b_I (x_{i_1} \wedge \cdots \wedge x_{i_k})^*.$$

a. Prove that

$$\langle \alpha, \beta \rangle = (\alpha \wedge \beta^*)^*$$

defines an inner product on $\mathcal{A}^k(\mathbb{R}^n)$ which makes the elementary k forms an orthonormal basis.

b. Given an $n \times k$ matrix M with column vectors $v_1, ..., v_k \in \mathbb{R}^n$, with entries $v_i = \sum_{j=1}^n m_{ji} e_j$, let the dual form be $\ell_i = \sum_{j=1}^n m_{ji} x_j$. Let
$$\omega = \ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_k.$$
Prove that
$$V(M)^2 = \langle \omega, \omega \rangle = (\omega \wedge \omega^*)^*,$$
where $V(M)$ denotes the k-dimensional volume of the parallelpiped spanned by the column vectors of M.

Problem 7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map, which acts on the standard basis $e_1, ..., e_n$ by
$$Te_i = \sum_{j=1}^n m_{ij} e_j.$$
Let $x_1, ..., x_n$ be the dual basis to $e_1, ..., e_n$. Prove
$$T^*(x_1 \wedge \cdots \wedge x_n) = \det(m_{ij}) x_1 \wedge \cdots \wedge x_n.$$