MATH 322, SPRING 2019 FINAL, PRACTICE PROBLEMS

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Problem 1. Let $\alpha : (\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$ be the spherical coordinate map. Let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Calculate $\alpha^*\omega$.

Problem 2. Let $\omega = e^{xyz}dx \wedge dy + x^3ydx \wedge dz + \sin(z)dx \wedge dy$. Calculate $d\omega$.

Problem 3. Let C be the right circular cylinder $\{x^2 + y^2 \leq 1, 0 \leq z \leq 5\}$, given the usual orientation in $\mathbb{R}^3$. Let

$$\omega = e^z dx \wedge dy.$$

Calculate $\int_{\partial C} \omega$.

Problem 4. Let $\omega = \sum_{i=1}^n x_i dx_i$. Calculate $\int_C \omega$ where C is the oriented curve

$$C = \left\{\cos(2\pi t), \cos(4\pi t), ..., \cos(2n\pi t) : 0 \leq t \leq \frac{1}{4}\right\}.$$

Problem 5. Let $\mathbb{T}$ be the torus in $\mathbb{R}^4$, $\mathbb{T} = \{x_1^2 + x_2^2 = a^2, x_3^2 + x_4^2 = b^2\}$. Prove that $\mathbb{T}$ is orientable, and calculate its volume.

Problem 6. Let $\mathrm{tr} : \mathrm{Mat}_{n\times n} \to \mathbb{R}$, $\mathrm{tr}\, M = \sum_i M_{ii}$ be the trace map. Let O_n be the orthogonal group of $n \times n$ matrices. Calculate the moments

$$m_1 = \frac{\int_{O_n} \mathrm{tr}(M)dV}{\int_{O_n} 1dV}$$
$$m_2 = \frac{\int_{O_n} (\mathrm{tr}\, M)^2 dV}{\int_{O_n} 1dV}.$$

Problem 7. Let $S^{n-1} = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| = 1\}$. Let O_n be the orthogonal group of $n \times n$ matrices M_n which satisfy $M_n^t M_n = I_n$.

a. Given $\underline{x} \in S^{n-1}$, let $O_{\underline{x}}$ be the orthogonal matrix which rotates $\underline{e}_1$ to $\underline{x}$ in the $(\underline{e}_1, \underline{x})$ plane while preserving their orientation, while leaving the orthogonal complement of this plane fixed. Give a matrix representation of $O_{\underline{x}}$.

b. Identify the $(n-1) \times (n-1)$ orthogonal group O_{n-1} with matrices in O_n whose first row and column are $\underline{e}_1$. Prove that each matrix M_n in O_n has a unique representation as $O_{\underline{x}} M_{n-1}$ with $\underline{x} \in S^{n-1}$ and $M_{n-1} \in O_{n-1}$.

c. Using the described coordinates, or otherwise, calculate the volume of O_n. It may be helpful to use that the volume on O_n is invariant under left and right translation, which was proved in the problems for Midterm 2. This reduces to the case that $\underline{x}$ is a neighborhood of $\underline{e}_1$.

Problem 8. Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle. Let $\gamma, \delta : S^1 \to \mathbb{R}^3$ be non-intersecting smooth loops, so $\gamma(s) \neq \delta(t)$ for all s, t. Define $F_{\gamma,\delta}(s,t) = \gamma(s) - \delta(t)$. Let

$$\ell(\gamma, \delta) := \frac{1}{4\pi} \int_{S^1 \times S^1} F_{\gamma,\delta}^* \left(\frac{xdy \wedge dz + ydz \wedge dx + zdy \wedge dz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right)$$

be the linking number. Prove

a. $\ell(\gamma, \delta)$ is unchanged if γ and δ are continously deformed without intersecting.

b. $\ell(\gamma, \delta) \in \mathbb{Z}$.

c. If $\gamma(s) = (\cos s, \sin s, 0)$ for $s \in [0, 2\pi]$ and $\delta(t) = (-1 + \frac{1}{2}\cos t, 0, \frac{1}{2}\sin t)$ for $t \in [0, 2\pi]$, then $\ell(\gamma, \delta) = 1$.