MATH 320, FALL 2017 PRACTICE FINAL EXAM

DECEMBER 15

Each problem is worth 10 points.

Problem 1.

a. (4 points) State the definition of a sequence of functions on an interval $[a, b]$ which converges uniformly to a function f.

b. (6 points) Prove that a sequence of functions $\{f_n\}$ on an interval $[a, b]$ which is uniformly Cauchy converges uniformly to a limit function f on $[a, b]$.

Solution.

a. $\{f_n\}$ converges uniformly to f if, for all $\epsilon > 0$ there exists N such that $n > N$ implies

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon.$$

b. For a fixed $x \in [a, b]$,

$$|f_n(x) - f_m(x)| \leq \sup_{y \in [a,b]} |f_n(y) - f_m(y)|,$$

so the sequence of real numbers $\{f_n(x)\}$ is Cauchy, and hence converges to a real number $f(x)$. To check that the pointwise convergence is in fact uniform, given $\epsilon > 0$, let N be such that $m, n > N$ implies that for all $x \in [a, b]$,

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2}.$$

Let $n \to \infty$ to conclude that, for $m > N$ and for all $x \in [a, b]$,

$$|f_m(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$$

and hence $f_m \to f$ uniformly.

Problem 2. A function f on $[a,b]$ is said to be convex on $[a,b]$ if for any $a \le x < y \le b$ and for any $0 \le t \le 1$,

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y).$$

a. (4 points) Prove that if f is convex on $[a,b]$, then for any $a \le x < y \le z < w \le b$,
$$\frac{f(y)-f(x)}{y-x} \le \frac{f(w)-f(z)}{w-z}.$$
b. (6 points) Prove that if f is convex on $[a,b]$, then it is integrable there. (Hint: you may use, without proof, that an increasing function on an interval $[a,b]$ is integrable.)

Solution.

a. Assume $0 < t < 1$, $c < d$ and write the equation of convexity as
$$t\left(f(tc+(1-t)d)-f(c)\right) \le (1-t)\left(f(d)-f(tc+(1-t)d)\right),$$
or, equivalently,
$$\frac{f(tc+(1-t)d)-f(c)}{(1-t)(d-c)} \le \frac{f(d)-f(tc+(1-t)d)}{t(d-c)}.$$
Applying this with $t=\frac{z-y}{z-x}$, $c=x$, $d=z$ gives $tc+(1-t)d=y$, and thus
$$\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(y)}{z-y}.$$
Applying the inequality again, now with $c=y$, $d=w$ and $t=\frac{w-z}{w-y}$, gives $tc+(1-t)d=z$ and thus
$$\frac{f(z)-f(y)}{z-y} \le \frac{f(w)-f(z)}{w-z}.$$
Combining the two yields
$$\frac{f(y)-f(x)}{y-x} \le \frac{f(w)-f(z)}{w-z}.$$

b. We first check that f is bounded on $[a, b]$. By convexity, for $x \in [a, b]$, $f(x) \leq \max(f(a), f(b))$ so f is bounded above. To prove that f is bounded below, note that the graph of f on $[a, \frac{a+b}{2}]$ is bounded below by the line through $(\frac{a+b}{2}, f(\frac{a+b}{2}))$ and $(b, f(b))$, while for $x \in [\frac{a+b}{2}, b]$, the graph of f is bounded below by the line through $(a, f(a))$ and $(\frac{a+b}{2}, f(\frac{a+b}{2}))$. Hence f is bounded.

Let $M > 0$ be such that $|f(x)| \leq M$ for $x \in [a, b]$. Given $\epsilon > 0$, let $a_0 = \min(a + \frac{\epsilon}{6M}, b)$ and $b_0 = \max(a_0, b - \frac{\epsilon}{6M})$. Since any pair of upper and lower Darboux sums differ by at most $\frac{\epsilon}{3}$ on $[a, a_0]$ and on $[b_0, b]$, it suffices to check that f is integrable on $[a_0, b_0]$, which we may assume is a non-degenerate interval. For $a_0 \leq x < y \leq b_0$,

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(a_0) - f(a)}{a_0 - a} := L.$$

Hence $F(x) = f(x) - Lx$ is increasing on $[a_0, b_0]$, hence integrable there. It follows that $f(x)$ is integrable there, also.

Problem 3.

a. (7 points) A sequence $\{a_n\}_{n=0}^{\infty}$ is defined recursively by

$$a_0 = 0, \qquad a_1 = 1$$
$$a_{n+1} = 5a_n - 6a_{n-1}, \qquad n \geq 1.$$

Define $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Find a closed form expression for $f(x)$ and determine it's radius of convergence.

b. (3 points) Determine, with proof, the value a_{1000}.

Solution.

a. It's straightforward to check by induction that, for all n, $|a_n| \leq 11^n$. Hence the series has radius of convergence at least $\frac{1}{11}$ and converges absolutely for $|x| < \frac{1}{11}$. For these x, applying the recurrence relation and justifying the manipulations by absolute convergence,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n x^n \\ &= x + \sum_{n=2}^{\infty} (5a_{n-1} - 6a_{n-2}) x^n \\ &= x + 5xf(x) - 6x^2 f(x) \end{aligned}$$

and hence $(6x^2 - 5x + 1)f(x) = x$ or

$$f(x) = \frac{x}{(2x-1)(3x-1)} = \frac{1}{1-3x} - \frac{1}{1-2x}.$$

Since the power series for $f(x)$ and the geometric series $\frac{1}{1-3x}$ and $\frac{1}{1-2x}$ all have radius of convergence at least $\frac{1}{11}$ about 0, differentiating n times and setting $x = 0$ determines the coefficients term-by-term, so $a_n = 3^n - 2^n$. Thus the full radius of convergence of $f(x)$ is $\frac{1}{3}$.

b. $3^{1000} - 2^{1000}$.

Problem 4.

a. (4 points) Let f be defined on $[0,1]$ by $f(x) = 1$ if x is rational, $f(x) = 0$ otherwise. Prove that f is not Riemann integrable.

b. (6 points) Let f be defined on $[0,1]$ by $f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ is rational in lowest terms, $f(x) = 0$ otherwise. Prove that $\int_0^1 f(x)dx = 0$.

Solution.

a. It follows from the Archimedean property of the real numbers that every non-empty interval contains both rational and irrational numbers. Hence, for any partition of $[0,1]$ the upper Darboux integral is 1 while the lower Darboux integral is 0, so f is not integrable.

b. The number of rationals in $[0,1]$ with denominator in lowest terms at most q is no more than $q^2 + 1$. For fixed q, form a partition of $[0,1]$ by including the endpoints of each interval of length $\frac{1}{2^q}$ centered at each rational in lowest terms of denominator at most q. Outside these intervals, f is bounded by $\frac{1}{q}$, so that the upper Darboux integral is bounded by $\frac{q^2+1}{2^q} + \frac{1}{q}$. Note that the lower Darboux integral is 0 for all partitions by arguing as in part a. Letting $q \to \infty$, $\frac{q^2+1}{2^q} + \frac{1}{q} \to 0$, so the integral exists and is 0.

Problem 5. Determine the following limits.

a. (5 points)

$$\lim_{x\to 0} \frac{\cos x - 1}{x - \log(1+x)}.$$

b. (5 points)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{1}{1 + \left(\frac{j}{N}\right)^2}.$$

Solution.

a. Since $\cos 0 = 1$ and $\log 1 = 0$ the limit is indeterminant of type $\frac{0}{0}$. Applying l'Hospital, differentiating top and bottom yields $\lim_{x\to 0} \frac{-\sin x}{1-\frac{1}{1+x}}$. Again, $\sin 0 = 0$ the denominator has value 0 at 0 also. Hence the limit is indeterminant of type $\frac{0}{0}$, so applying l'Hospital one further time yields

$$\lim_{x\to 0} \frac{-\cos x}{\frac{1}{(1+x)^2}} = -1.$$

b. The function $\frac{1}{1+x^2}$ is continuous, hence integrable on $[0,1]$. The object in the limit is the upper Darboux sum for a partition of $[0,1]$ into N equal size intervals, and hence, since the mesh size tends to 0, the limit is the integral, which is

$$\int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4}.$$

Problem 6.

a. (4 points) Find the degree 3 Taylor polynomial of e^{e^x-1} about $x = 0$.
b. (6 points) Prove that the radius of convergence of the Taylor series for e^{e^x-1} is at least 1.

Solution.

a. Write, using Taylor expansion, $e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$ where $O(x^4)$ denotes a quantity bounded in size by a constant times x^4 as $x \to 0$. Then $(e^x-1)^2 = x^2 + x^3 + O(x^4)$ and $(e^x-1)^3 = x^3 + O(x^4)$. Note that $u = e^x - 1$ tends to 0 as $x \to 0$ and $e^x - 1 = O(x)$ as $x \to 0$. Hence,

$$\begin{aligned} e^u &= 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + O(u^4) \\ &= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{1}{2}(x^2 + x^3) + \frac{1}{6}x^3 + O(x^4) \\ &= 1 + x + x^2 + \frac{5}{6}x^3 + O(x^4) \end{aligned}$$

so the degree 3 Taylor polynomial is $1 + x + x^2 + \frac{5}{6}x^3$.

b. Let $P_n(u)$ be a sequence of polynomials such that

$$\left(\frac{d}{dx}\right)^n \left(e^{e^x-1}\right) = e^{e^x-1} P_n(e^x).$$

Differentiating, using the chain and product rules, proves the recurrence relation

$$P_0(u) = 1, \qquad P_{n+1}(u) = u(P_n(u) + P_n'(u)).$$

The Taylor series of e^{e^x-1} about 0 is

$$\sum_{n=0}^{\infty} \frac{P_n(1)}{n!} x^n.$$

The recurrence condition guarantees that P_n is a polynomial with non-negative integer coefficients, of degree at most n. We check by induction that, for all n, the sum of the coefficients of P_n, which is equal to $P_n(1)$, is at most $n!$, so that the radius of convergence is at least 1. Indeed, $P_0(1) = 1 = 0!$, and for $n \geq 0$, $P_n'(1) \leq nP_n(1)$, so

$$P_{n+1}(1) = P_n(1) + P_n'(1) \leq (n+1)P_n(1) \leq (n+1)!$$

by invoking the inductive assumption.