MATH 307, FALL 2020 PRACTICE MIDTERM 1 SOLUTIONS

SEPTEMBER 28

Each problem is worth 10 points.

Problem 1. Determine the distance between $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ and the line $\ell(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Solution. The distance is equal to the distance from $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ to the line through 0 with direction $u = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix}$. The component of $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ in direction u is $\begin{pmatrix} \frac{12}{5} \\ 0 \\ \frac{6}{5} \end{pmatrix}$, and the component orthogonal is $\begin{pmatrix} \frac{3}{5} \\ -1 \\ \frac{-6}{5} \end{pmatrix}$. Thus the distance to the line is $d^2 = \frac{9}{25} + 1 + \frac{36}{25}$ or $d = \sqrt{\frac{14}{5}}$.

Problem 2. Express $u = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ as a component parallel to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and a component perpendicular.

Solution. Let $w = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Thus the component in the direction of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $(u \cdot w)w = \begin{pmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}$ and the component orthogonal is $\begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$.

Problem 3. For each of the linear systems below, express the system in row-reduced form and give the solution set.

a.

$$\begin{aligned} x + 2z &= 7 \\ 2x + y + z &= 3 \\ y + 2z &= 1. \end{aligned}$$

b.

$$\begin{aligned} x + y + z &= 10 \\ 2x + y &= 7. \end{aligned}$$

Solution. a. The following systems are equivalent

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right) \equiv \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -3 & -11 \\ 0 & 1 & 2 & 1 \end{array}\right)$$
$$\equiv \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -3 & -11 \\ 0 & 0 & 5 & 12 \end{array}\right)$$
$$\equiv \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -3 & -11 \\ 0 & 0 & 1 & \frac{12}{5} \end{array}\right)$$
$$\equiv \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{5} \\ 0 & 1 & 0 & -\frac{19}{5} \\ 0 & 0 & 1 & \frac{12}{5} \end{array}\right).$$

Hence $x = \frac{11}{5}, y = \frac{-19}{5}, z = \frac{12}{5}$.

b. The following systems are equivalent

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & 1 & 0 & 7 \end{array}\right) \equiv \left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -1 & -2 & -13 \end{array}\right)$$
$$\equiv \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 13 \end{array}\right).$$

Hence the solution set is

$$\left\{ \begin{pmatrix} z-3 \\ -2z+13 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}.$$

Problem 4.

a. Calculate the inverse matrix of

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

b. Calculate the determinant

$$\det \begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. a. The following matrices are equivalent

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \equiv \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\equiv \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right).$$

Thus the inverse is $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ -1 & 1 & 1 \end{pmatrix}$.

b. We have

$$\det \begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = -2.$$

Problem 5. Find the equation of the plane through

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}.$$

Solution. The two vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\0\end{pmatrix}$ are parallel to the plane, so their cross product

$$n = \det\begin{pmatrix}\underline{i} & \underline{j} & \underline{k}\\1 & 1 & 1\\-1 & 3 & 0\end{pmatrix} = \begin{pmatrix}-3\\-1\\4\end{pmatrix}$$

is orthogonal to it. Thus the plane has equation

$$-3x - y + 4z = -3.$$

Problem 6. Find the equation of the plane tangent to $2x^2 + 2y^2 - z^2 = 12$ at $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

Solution. Let $f(x, y, z) = 2x^2 + 2y^2 - z^2 - 12$. Then $\nabla f = \begin{pmatrix} 4x \\ 4y \\ -2z \end{pmatrix}$. Thus

$$\nabla f(2, 2, 2) = \begin{pmatrix} 8 \\ 8 \\ -4 \end{pmatrix}$$

so the equation of the tangent plane is $8x + 8y - 4z = 24$.