MATH 307, FALL 2020 MIDTERM 2 SOLUTIONS

OCTOBER 26

Each problem is worth 10 points.

Problem 1. Find all critical points of $f(x, y) = x^4 + xy + y^2$ and determine if each is a max, min, or saddle point.

Solution. At a critical point

$$\nabla f(x, y) = \begin{pmatrix} 4x^3 + y \\ x + 2y \end{pmatrix} = 0.$$

Thus $y = -\frac{x}{2}$ and $4x^3 - \frac{x}{2} = 0$ so either $x = 0$, $y = 0$ or $x = \pm\frac{\sqrt{2}}{4}$, $y = \mp\frac{\sqrt{2}}{8}$. The Hessian is

$$H_f = \begin{pmatrix} 12x^2 & 1 \\ 1 & 2 \end{pmatrix}$$

so that $D = 24x^2 - 1$. This is negative at $(0, 0)$, which is a saddle point, and positive at $(\pm\frac{\sqrt{2}}{4}, \mp\frac{\sqrt{2}}{8})$, which is a local minimum since $f_{xx} > 0$.

Problem 2. Find the point closest to $\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ on $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

Solution. Let R be the rotation about the origin which carries $\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ to $\begin{pmatrix} \sqrt{62} \\ 0 \\ 0 \end{pmatrix}$. The closest point to this point on the sphere is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and hence, by rotational symmetry, the closest point on the sphere to $\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ is $\frac{1}{\sqrt{62}} \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$.

Problem 3. Determine whether each limit exists.

a.
$$\lim_{(x,y)\to(0,0)} \frac{x^3 - 3x^2y + y^3}{x^2 + y^2}.$$

b.
$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}.$$

Solution.

a. When $x^2 + y^2 = \delta$, $|x^3 - 3x^2y + y^3| \leq 5\delta^{\frac{3}{2}}$, and hence the limit is 0.

b. When $y = 0$, $\lim_{x\to 0} \frac{x^2-2y^2}{x^2+y^2} = 1$, while when $x = 0$, $\lim_{y\to 0} \frac{x^2-2y^2}{x^2+y^2} = -2$. Since the limits are not equal, the limit as $(x, y) \to (0, 0)$ together does not exist.

Problem 4. Find a vector normal to the surface $xyz = 1000$ at $\begin{pmatrix} 20 \\ 5 \\ 10 \end{pmatrix}$. Find the equation of a tangent plane at the point. In a neighborhood of the point, z is a function of x and y. Find z_x and z_y.

Solution. Let $g(x, y, z) = xyz$. Then $\nabla g(x, y, z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \begin{pmatrix} 50 \\ 200 \\ 100 \end{pmatrix}$, which is a normal vector. Thus the tangent plane has equation

$$50(x - 20) + 200(y - 5) + 100(z - 10) = 0.$$

We have $z = \frac{1000}{xy}$. Thus

$$z_x = -\frac{1000}{x^2 y}, \qquad z_y = -\frac{1000}{xy^2}.$$

Problem 5. Let $f(u,v) = \begin{pmatrix} u^3 - v^3 \\ u^3 + v^3 \\ 3u^2 v \end{pmatrix}$, $g(x,y,z) = \begin{pmatrix} xy \\ yz \end{pmatrix}$. Calculate f', g' and $(g \circ f)'$.

Solution. We have $f'(u,v) = \begin{pmatrix} 3u^2 & -3v^2 \\ 3u^2 & 3v^2 \\ 6uv & 3u^2 \end{pmatrix}$ and $g'(x,y,z) = \begin{pmatrix} y & x & 0 \\ 0 & z & y \end{pmatrix}$.

Thus

$$\begin{aligned}
(g \circ f)'(u,v) &= g'(f(u,v))f'(u,v) \\
&= \begin{pmatrix} u^3 + v^3 & u^3 - v^3 & 0 \\ 0 & 3u^2 v & u^3 + v^3 \end{pmatrix} \begin{pmatrix} 3u^2 & -3v^2 \\ 3u^2 & 3v^2 \\ 6uv & 3u^2 \end{pmatrix} \\
&= \begin{pmatrix} 6u^5 & -6v^5 \\ 15u^4 v + 6uv^4 & 12u^2 v^3 + 3u^5 \end{pmatrix}.
\end{aligned}$$

Problem 6. Find the derivative of $F(x,y,z) = \begin{pmatrix} x^2+y^2+z^2 \\ 2xyz \\ x^3+2z^3 \end{pmatrix}$. Calculate $F'(1,2,1)$ and $(F^{-1})'(6,4,3)$.

Solution. We have $F'(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ 2yz & 2xz & 2xy \\ 3x^2 & 0 & 6z^2 \end{pmatrix}$ so that

$$F'(1,2,1) = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 4 \\ 3 & 0 & 6 \end{pmatrix}.$$

It follows that

$$\left(F^{-1}\right)'(6,4,3) = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 4 \\ 3 & 0 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & 0 \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$