LAST CLASS I INTRODUCED AN INNER PRODUCT $\langle , \rangle$ ON A VECTOR SPACE, WHICH GENERALIZES THE DOT PRODUCT.

EX: IF $f, g \in C[0,1]$ = CTS FNS ON $[0,1]$.

THEN $\langle f, g \rangle = \int_0^1 f(x) \cdot g(x)\, dx.$

① $\int_0^1 |f|^2 dx \geq 0$, $= 0$ IFF $f = 0$.

(POSITIVITY)

② $\int_0^1 f(x) g(x)\, dx = \int_0^1 g(x) f(x)\, dx$

(SYMMETRY) SO

$\langle f, g \rangle = \langle g, f \rangle$

③ $\langle af + bg, h \rangle$

$= \int_0^1 (a f(x) + b g(x)) \cdot h(x)\, dx$

$= a \cdot \int_0^1 f(x) h(x)\, dx + b \int_0^1 g(x) h(x)\, dx$

$= a \langle f, h \rangle + b \langle g, h \rangle.$

ORTHONORMAL BASES:

THE STANDARD BASIS VECTORS OF $\mathbb{R}^n$ FORM AN ORTHONORMAL SET BECAUSE

$$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & \text{IF } i=j \\ 0 & \text{OTHERWISE.} \end{cases}$$

THEY ARE UNIT LENGTH, PAIRWISE PERPENDICULAR.

PROOF: SINCE $\underline{u}_1, \dots, \underline{u}_n$ HAS n VECTORS, IT SUFFICES TO CHECK THAT THE SET IS LINEARLY INDEPENDENT TO SHOW THAT IT IS A BASIS.

SUPPOSE $\underline{0} = c_1\underline{u}_1 + c_2\underline{u}_2 + \dots + c_n\underline{u}_n$.

IS A LINEAR RELATION.

$$0 = \langle \underline{0}, \underline{u}_i \rangle = \langle c_1\underline{u}_1 + \dots + c_n\underline{u}_n, \underline{u}_i \rangle$$
$$= c_1\langle \underline{u}_1, \underline{u}_i \rangle + c_2\langle \underline{u}_2, \underline{u}_i \rangle + \dots + c_n\langle \underline{u}_n, \underline{u}_i \rangle$$

$$= c_i$$

SO $0 = c_i$ FOR ALL i

$\Rightarrow$ SET IS LINEARLY INDEP.

EXAMPLE: CONSIDER

$C[0,1]$ = CTS FUNCTIONS ON $[0,1]$.

THE SUBSPACE OF TRIGONOMETRIC POLYNOMIALS $\mathcal{J}$ IS THE SPAN OF

$$\{1, \sin(2\pi x), \sin(4\pi x), \sin(6\pi x), \ldots, \cos(2\pi x), \cos(4\pi x), \ldots\}.$$

IN THIS SET

$$\langle 1, 1\rangle = \int_0^1 1\,dx = 1.$$

$$\langle 1, \sin 2k\pi x\rangle = \int_0^1 \sin(2\pi kx)\,dx = 0$$

$$\langle 1, \cos 2\pi kx\rangle = \int_0^1 \cos(2\pi kx)\,dx = 0$$

$$\langle \sin 2\pi kx, \cos 2\pi jx\rangle = \int_0^1 \sin 2\pi kx \cos 2\pi jx\,dx = 0.$$

$$\langle \sin 2\pi kx, \sin 2\pi jx\rangle = \int_0^1 \sin 2\pi kx \sin 2\pi jx\,dx = \begin{cases} \frac{1}{2} & \text{IF } k=j \\ 0 & \end{cases}$$

$$\langle \cos 2\pi kx, \cos 2\pi jx\rangle = \int_0^1 \cos 2\pi kx \cos 2\pi jx\,dx = \begin{cases} \frac{1}{2} & k=j \\ 0 & \end{cases}.$$

THUS $\left\{1, \frac{\sin 2\pi kx}{\sqrt{2}}, \frac{\cos 2\pi jx}{\sqrt{2}}\right\}$

IS AN ORTHONORMAL SET.

EULER'S FORMULA

$$\cos 2\pi x = \frac{e^{2\pi i x} + e^{-2\pi i x}}{2}.$$

$$\sin 2\pi x = \frac{e^{2\pi i x} - e^{-2\pi i x}}{2i}.$$

$$\int_0^1 e^{2\pi i k x}\,dx = \begin{cases} 0 & \text{IF } k \neq 0 \\ 1 & \text{IF } k = 0. \end{cases}$$

$\{e^{2\pi i k x} : k \in \mathbb{Z}\}$ O.N. BASIS FOR COMPLEX CTS FNS ON $[0,1]$.

<u>REMARK:</u> AN ORTHONORMAL SET S IN AN INNER PRODUCT SPACE V IS CALLED <u>COMPLETE</u> IF,

FOR ALL $\underline{x} \in V$,

$$\|\underline{x}\|^2 = \sum_{\underline{y} \in S} \langle \underline{x}, \underline{y} \rangle^2 .$$

PYTHAGOREAN THEOREM:

IF V IS AN INNER PRODUCT SPACE WITH ORTHONORMAL BASIS $\{\underline{u}_1, \dots, \underline{u}_n\}$, FOR ALL $\underline{x} \in V$,

$$\|\underline{x}\|^2 = \sum_{k=1}^{n} \langle \underline{x}, \underline{u}_k \rangle^2$$

AND $\underline{x} = \sum_{k=1}^{n} \langle \underline{x}, \underline{u}_k \rangle \underline{u}_k$.

THIS FOLLOWS FROM THE PREVIOUS THEOREM.

LET $\underline{u}_j = \underline{x}_j + \underline{y}_j$ WHERE $\underline{x}_j$ IS THE VECTOR IN THE SPAN OF $\underline{u}_1, \dots, \underline{u}_{j-1}$ MINIMIZING THE DISTANCE. LET $\underline{v}_j = \frac{\underline{y}_j}{\|\underline{y}_j\|}$.

NOTICE $\underline{y}_j \neq 0$ SINCE $\underline{u}_j$ IS NOT IN SPAN $\{\underline{u}_1, \dots, \underline{u}_{j-1}\}$.

BY THE PREVIOUS THEOREM, IF SPAN $\{\underline{v}_1, \dots, \underline{v}_{j-1}\}$ $= \text{SPAN}\{\underline{u}_1, \dots, \underline{u}_{j-1}\}$

AND $\underline{v}_1, \dots, \underline{v}_{j-1}$ O.N.

THEN $\underline{x}_j = \sum_{i=1}^{j-1} \langle \underline{u}_j, \underline{v}_i \rangle \cdot \underline{v}_i$

VECTOR IN SPAN MINIMIZING DISTANCE.

$$\underline{y}_j = \underline{u}_j - \underline{x}_j$$
$$= \underline{u}_j - \sum_{i=1}^{j-1} \langle \underline{u}_j, \underline{v}_i \rangle \underline{v}_i .$$

WE HAVE

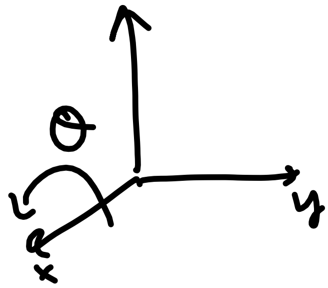
$$\langle \underline{y}_j, \underline{v}_i \rangle = \left\langle \underline{u}_j - \sum_{\ell=1}^{j-1} \langle \underline{u}_j, \underline{v}_\ell \rangle \underline{v}_\ell, \underline{v}_i \right\rangle$$
$$= \langle \underline{u}_j, \underline{v}_i \rangle - \sum_{\ell=1}^{j-1} \langle \underline{u}_j, \underline{v}_\ell \rangle \langle \underline{v}_\ell, \underline{v}_i \rangle$$
$$= \langle \underline{u}_j, \underline{v}_i \rangle - \langle \underline{u}_j, \underline{v}_i \rangle = 0.$$

THUS $\underline{y}_j$, AND ALSO $\underline{v}_j$, IS $\perp$ TO $\underline{v}_1, \dots, \underline{v}_{j-1}$. $\|\underline{v}_j\| = 1$.

REMARK REGARDING CHANGE OF BASIS:

THE MATRIX

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$



MAKES A ROTATION OF θ ABOUT THE X-AXIS.

GIVEN AN ORTHONORMAL (POS. ORIENTED BY RIGHT HAND RULE) BASIS $\underline{u}_1, \underline{u}_2, \underline{u}_3$ WE CAN

$\underline{u}_3 = \underline{u}_1 \times \underline{u}_2$.

ROTATE BY θ ABOUT $\underline{u}_1$ BY FORMING THE MATRIX

$$\begin{pmatrix} | & | & | \\ \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} - & \underline{u}_1 & - \\ - & \underline{u}_2 & - \\ - & \underline{u}_3 & - \end{pmatrix}.$$

THIS IS BECAUSE

$$\begin{pmatrix} - & \underline{u}_1 & - \\ - & \underline{u}_2 & - \\ - & \underline{u}_3 & - \end{pmatrix} \begin{pmatrix} | & | & | \\ \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

SO $\begin{pmatrix} | & | & | \\ \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ | & | & | \end{pmatrix}^{-1} = \begin{pmatrix} - & \underline{u}_1 & - \\ - & \underline{u}_2 & - \\ - & \underline{u}_3 & - \end{pmatrix}$.

<u>DERIVATIVES:</u> A FUNCTION

$f: D \to \mathbb{R}^n, \quad D \subset \mathbb{R}^m$

HAS <u>COORDINATE FUNCTIONS</u>

$$f(\underline{x}) = \begin{bmatrix} f_1(\underline{x}) \\ \vdots \\ f_n(\underline{x}) \end{bmatrix}$$

DEF'N:
THE LIMIT AS $t \to a$ OF A VECTOR VALUED FUNCTION $\underline{f}(t)$ IS

$$\lim_{t \to a} \underline{f}(t) = \begin{pmatrix} \lim_{t \to a} f_1(t) \\ \vdots \\ \lim_{t \to a} f_n(t) \end{pmatrix}.$$

THE <u>DERIVATIVE</u> OF $\underline{f}(t)$

IS

$$\lim_{h \to 0} \frac{\underline{f}(t+h) - \underline{f}(t)}{h} = \begin{pmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{pmatrix}.$$

THE TANGENT LINE
TO THE CURVE AT $g(t_0)$
HAS EQUATION

$$\underline{t}(t) = \underline{g}(t_0) + (t - t_0)\underline{g}'(t_0).$$

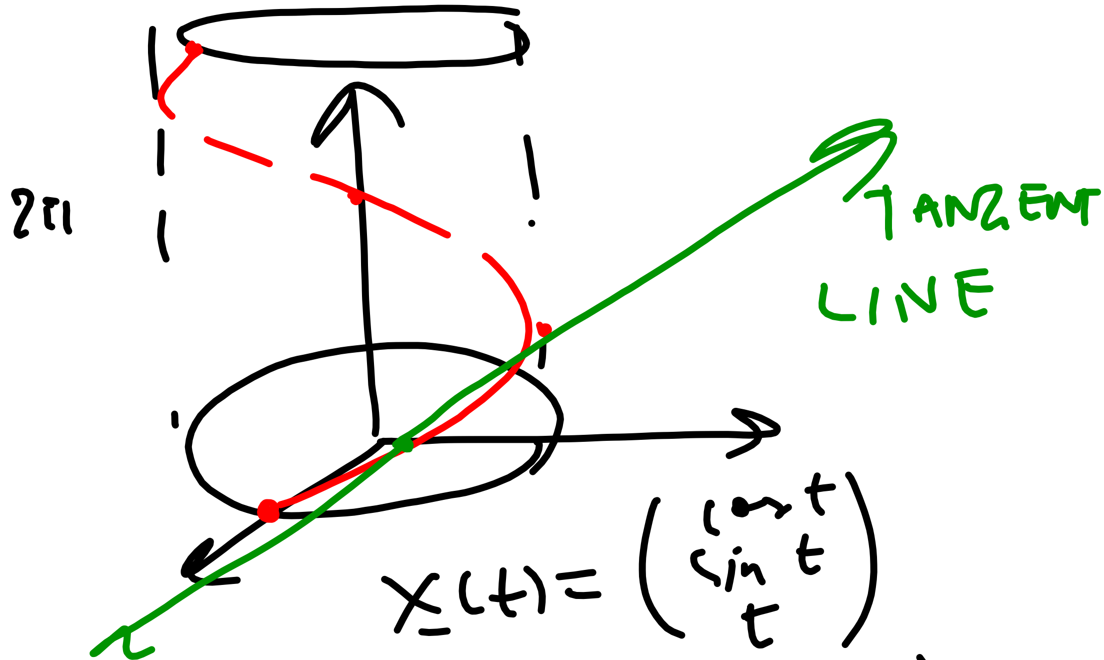
IN PHYSICS ITS USUAL TO DENOTE:

$\underline{x}(t)$ = POSITION AT TIME t

$\underline{v}(t) = \underline{x}'(t)$ VELOCITY.

$\|\underline{v}(t)\|$ = SPEED

$\underline{a}(t) = \underline{v}'(t)$ ACCELERATION



$$\underline{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

$$\underline{v}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$