MAT 307 LECTURE 9

MIDTERM 1 IS MONDAY

COVERS THROUGH THIS LECTURE

IN A GENERAL INNER PRODUCT SPACE, IF $f, g \neq 0$, DEFINE THE ANGLE BETWEEN f, g BY

$$\cos\theta = \frac{\langle f, g\rangle}{\|f\|\cdot\|g\|}.$$

($-1 \leq \frac{\cdots}{\cdots} \leq 1$ BY CAUCHY-SCHWARZ)

AN INNER PRODUCT SPACE IS "GEOMETRIC" SINCE THERE IS A NOTION OF ANGLES BTWN VECTORS.

<u>THEOREM</u>: SUPPOSE $\mathcal{V}$ IS AN n-DIM'L VECTOR SPACE, $\mathcal{B} = \{\underline{u}_1, \dots, \underline{u}_n\}$ IS AN ORTHONORMAL SET. THEN $\mathcal{B}$ IS A <u>BASIS</u> FOR $\mathcal{V}$, AND IF $\underline{x} = u_1 \underline{u}_1 + \dots + u_n \underline{u}_n$ THEN

$$u_j = \langle \underline{x}, \underline{u}_j \rangle.$$

THIS PROVES THAT THE SET IS A BASIS, AND VERIFIES

$\underline{x} = c_1 \underline{u}_1 + \ldots + c_n \underline{u}_n,$

THEN $\langle \underline{x}, \underline{u}_i \rangle = c_i$. $\square$

THEOREM: LET $S=\{\underline{u}_1,\dots,\underline{u}_n\}$ BE AN ORTHONORMAL SET IN A VECTOR SPACE V. FOR ALL $\underline{x}\in V$, THE DISTANCE

$$\|\underline{x}-(u_1\underline{u}_1+\dots+u_n\underline{u}_n)\|^2$$

IS MINIMIZED BY CHOOSING $u_i=\langle\underline{x},\underline{u}_i\rangle$.

$$\|\underline{x}\|^2\geq\sum_{i=1}^{n}\langle\underline{x},\underline{u}_i\rangle^2.$$

"BESSEL'S INEQUALITY".

THE DISTANCE OF $\underline{x}$ FROM SPAN $\{\underline{u}_1,\dots,\underline{u}_n\}$ IS

$$d^2=\|\underline{x}\|^2-\sum_{i=1}^{n}\langle\underline{x},\underline{u}_i\rangle^2.$$

PROOF:

$$\|\underline{x} - (u_1 \underline{u}_1 + \dots + u_n \underline{u}_n)\|^2$$
$$= \langle \underline{x} - (u_1 \underline{u}_1 + \dots + u_n \underline{u}_n), \underline{x} - (u_1 \underline{u}_1 + \dots + u_n \underline{u}_n) \rangle$$

$$= \|\underline{x}\|^2 + \|u_1 \underline{u}_1 + \dots + u_n \underline{u}_n\|^2 - 2 \langle \underline{x}, u_1 \underline{u}_1 + \dots + u_n \underline{u}_n \rangle$$

$$\|u_1 \underline{u}_1 + \dots + u_n \underline{u}_n\|^2 = \langle u_1 \underline{u}_1 + \dots + u_n \underline{u}_n, u_1 \underline{u}_1 + \dots + u_n \underline{u}_n \rangle = \sum_{i=1}^{n} u_i^2 .$$

THUS $\|\underline{x} - (u_1 \underline{u}_1 + \dots + u_n \underline{u}_n)\|^2$
$$= \|\underline{x}\|^2 - 2 \sum_{k=1}^{n} u_k \langle \underline{x}, \underline{u}_k \rangle + \sum_{k=1}^{n} u_k^2 .$$

WRITE: $-2 u_k \langle \underline{x}, \underline{u}_k \rangle + u_k^2$
$$= (\langle \underline{x}, \underline{u}_k \rangle - u_k)^2 - \langle \underline{x}, \underline{u}_k \rangle^2$$

THUS: $\|\underline{x} - (u_1 \underline{u}_1 + \dots + u_n \underline{u}_n)\|^2$
$$= \|\underline{x}\|^2 - \sum_{k=1}^{n} \langle \underline{x}, \underline{u}_k \rangle^2 + \sum_{k=1}^{n} (u_k - \langle \underline{x}, \underline{u}_k \rangle)^2$$

$(*)$

THE MINIMIZING CHOICE SETS THE COEFFICIENT $u_k = \langle \underline{x}, \underline{u}_k \rangle$

SO $(*) = 0$.

THUS THE MINIMIZING DISTANCE IS OBTAINED BY

$$\sum_{k=1}^{n} \langle \underline{x}, \underline{u}_k \rangle \underline{u}_k \quad \text{WITH}$$

DISTANCE SQUARED

$$\|\underline{x}\|^2 - \sum_{k=1}^{n} \langle \underline{x}, \underline{u}_k \rangle^2 .$$

GRAM-SCHMIDT PROCESS:

GIVEN LINEARLY INDEPENDENT VECTORS $\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4, \ldots,$ AN ORTHONORMAL SEQUENCE $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \ldots$ WITH, FOR EACH k,

$$\text{SPAN}\{\underline{u}_1, \ldots, \underline{u}_k\} = \text{SPAN}\{\underline{v}_1, \ldots, \underline{v}_k\}$$

CAN BE OBTAINED AS FOLLOWS.

IT REMAINS TO CHECK THAT

$\mathrm{SPAN}\{\underline{u}_1, \ldots, \underline{u}_k\} = \mathrm{SPAN}\{\underline{v}_1, \ldots, \underline{v}_k\}$

ALL k.

BY INDUCTION:

$\underline{v}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|}$ SO SPANS EQUAL.

ASSUMING $\mathrm{SPAN}\{\underline{v}_1, \ldots, \underline{v}_{k-1}\}$
$= \mathrm{SPAN}\{\underline{u}_1, \ldots, \underline{u}_{k-1}\}$,

$$\underline{v}_k = \frac{\underline{u}_k - \sum_{j=1}^{n} \langle \underline{u}_k, v_{j-1} \rangle v_{j-1}}{\| \cdots \quad \cdots \|}.$$

$\Rightarrow \underline{v}_k \in \mathrm{SPAN}\{\underline{u}_k, \underline{v}_1, \ldots, \underline{v}_{k-1}\}$

$\subset \mathrm{SPAN}\{\underline{u}_1, \ldots, \underline{u}_k\}$.

THUS $\{\underline{v}_1, \ldots, \underline{v}_n\}$ ARE k LINEARLY INDEP. VECTORS IN $\mathrm{SPAN}\{\underline{u}_1, \ldots, \underline{u}_n\}$, SO A BASIS FOR THIS SPACE. $\square$

PROJECTION IN THE DIRECTION OF $\underline{u}_1$ IS GIVEN BY

$$\underline{u}_1 \cdot \underline{u}_1^t \, \underline{x} = \langle \underline{u}_1, \underline{x} \rangle \cdot \underline{u}_1 .$$

REFLECTION IN THE PLANE $\perp$ TO $\underline{u}_1$ IS GIVEN BY

$$R_{\underline{u}_1}(\underline{x}) = \underline{x} - 2(\underline{u}_1 \cdot \underline{x})\underline{u}_1$$

THE MATRIX IS $\boxed{\underline{I} - 2\,\underline{u}_1 \underline{u}_1^t .}$

IF f IS A FUNCTION OF A SINGLE VARIABLE t,

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

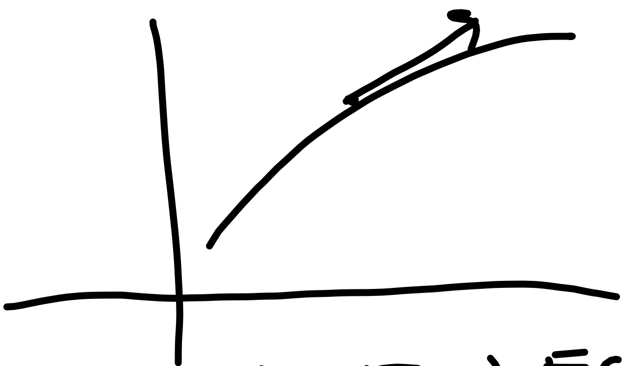
"VECTOR VALUED FUNCTION"

THE FUNCTION f IS <u>CTS</u>

IF EACH COMPONENT FUNCTION

IS <u>CONTINUOUS</u>.

THE IMAGE OF A VECTOR VALUED FUNCTION $g(t)$ IS CALLED A <u>CURVE</u>



THE <u>TANGENT VECTOR</u> TO THE CURVE AT $g(t)$ IS $g'(t)$.

DEFINITION: A CURVE WITH PARAMETRIC REPRESENTATION

$\underline{x}(t) = g(t)$

IS **SMOOTH** IF $g'(t)$

EXISTS FOR ALL t AND

IS NON-ZERO.

EXAMPLE: THE IMAGE OF THE CURVE

$$\underline{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

IS A HELIX, CONTAINED IN $x^2+y^2=1$

THE PRODUCT RULES OF 1-VARIABLE CALCULUS HAVE MULTI-DIMENSIONAL GENERALIZATIONS:

- IF $\phi(t)$ IS A SCALAR FN,

$$\frac{d}{dt}\left(\phi(t)\underline{x}(t)\right) = \phi'(t)\underline{x}(t) + \phi(t)\underline{x}'(t)$$

- IF $\underline{x}(t), \underline{y}(t)$ VECTOR VALUED FNS,

$$\frac{d}{dt}\left(\underline{x}(t)\cdot\underline{y}(t)\right) = \underline{x}'(t)\cdot\underline{y}(t) + \underline{x}(t)\cdot\underline{y}'(t)$$

$$\frac{d}{dt}\left(\underline{x}(t)\times\underline{y}(t)\right) = \underline{x}'(t)\times\underline{y}(t)$$
$$\underline{x}(t)\times\underline{y}'(t)$$

CHAIN RULE: $u: \mathbb{R}\to\mathbb{R}$

$$\frac{d}{dt}\underline{x}(u(t)) = \underline{x}'(u(t))\cdot u'(t).$$