MAT 307 LECTURE 7

BASES, DIMENSION

PRACTICE MIDTERM 1 ON THE COURSE WEBSITE

RECALL FROM LAST LECTURE:

EXAMPLES OF ABSTRACT VECTOR SPACES:

(1) POLYNOMIALS IN A VARIABLE X

(2) CONTINUOUS FUNCTIONS ON $[0,1]$.

(3) ALL FUNCTIONS $f:[0,1]\to\mathbb{R}$.

E.G: (1) MOVE IN DIRECTION $1, X, X^2, X^3, \ldots$
THIS SPACE IS ∞-DIM'L.

<u>THEOREM</u>: A LINEAR FUNCTION f IS 1-1 (OR INJECTIVE) IFF $\underline{x} = \underline{0}$ IS THE ONLY SOLUTION OF $f(\underline{x}) = \underline{0}$.

EXAMPLE: IF A IS A FIXED $m \times n$ MATRIX AND

$\mathcal{M}_{n,p}$, $\mathcal{M}_{m,p}$ ARE $n \times p$, $m \times p$ MATRICES.

$\mathcal{M}_{n,p}$, $\mathcal{M}_{m,p}$ ARE VECTOR SPACES.

THERE IS A LINEAR FUNCTION

$$f_A : \mathcal{M}_{n,p} \to \mathcal{M}_{m,p}$$

$$f_A(M) = A \cdot M.$$

PROOF:

$$g\circ f(a\underline{u}+b\underline{v})$$
$$= g(f(a\underline{u}+b\underline{v}))$$
$$= g(af(\underline{u})+bf(\underline{v}))$$
$$= a\cdot g(f(\underline{u}))+b\cdot g(f(\underline{v})). \quad \square$$

<u>EX:</u> THE INTEGRAL

$$\int_0^1$$

IS A LINEAR MAP $C[0,1] \to \mathbb{R}$

CTS FNS ON $[0,1]$.

THE DERIVATIVE IS A LINEAR MAP $C^1[0,1] \to C^0[0,1]$

CTSLY DIFF'L FNS ON $[0,1]$

CTS FNS ON $[0,1]$.

<u>PROOF</u>: SUPPOSE $\underline{x}, \underline{y} \in f(\mathcal{V})$

SO THERE EXIST $\underline{u}, \underline{v} \in \mathcal{V}$

WITH $f(\underline{u}) = \underline{x}$, $f(\underline{v}) = \underline{y}$.

THEN BY LINEARITY,

$$f(a\underline{u} + b\underline{v}) = a\underline{x} + b\underline{y} \in f(\mathcal{V})$$

SO $f(\mathcal{V})$ IS A SUBSPACE.

IF f IS 1-1,

$$f^{-1}(a\underline{x} + b\underline{y}) = a\underline{u} + b\underline{v}$$
$$= a f^{-1}(\underline{x}) + b f^{-1}(\underline{y}).$$

SO f^{-1} IS LINEAR. $\square$

<u>DEFINITION</u>: GIVEN A LINEAR MAP

$f : \mathcal{V} \to \mathcal{W}$

THE <u>NULL SPACE</u> IS

$$\mathcal{N} = \{ \underline{v} \in \mathcal{V} : f(\underline{v}) = \underline{0} \}.$$

THEOREM: LET

$f: V \to W$

BE A LINEAR MAP.

THEN $\mathrm{NULL}(f)$ IS A SUBSPACE OF V.

THEOREM: $f: V \to W$ LINEAR IS 1-1 IFF $\mathrm{NULL}(f) = \{\underline{0}\}$.

PF: WE ALREADY CHECKED

1-1 $\iff$ $f(\underline{x}) = \underline{0} \implies \underline{x} = \underline{0}$.

$\iff$ $\mathrm{NULL}(f) = \{\underline{0}\}$. $\square$

<u>DEFINITION:</u> WE SAY $\mathcal{S} \subset \mathcal{V}$ IS A <u>SPANNING SET</u> IF $\text{SPAN}(\mathcal{S}) = \mathcal{V}$.

<u>DEFINITION</u>: A <u>BASIS</u> FOR A VECTOR SPACE $\mathcal{V}$ IS A LINEARLY INDEPENDENT SET SPANNING $\mathcal{V}$.

<u>EX</u>: THE STANDARD BASIS OF $\mathbb{R}^n$ IS

$$\left\{ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}.$$

<u>THEOREM:</u> LET $\{\underline{b}_1, \ldots, b_n\}$ BE A BASIS FOR V. THEN FOR ANY $\underline{v} \in V$ THERE EXIST UNIQUE REAL NUMBERS $x_1, \ldots, x_n \in \mathbb{R}$ SUCH THAT

$$\underline{v} = x_1 \underline{b}_1 + x_2 \underline{b}_2 + \cdots + x_n \underline{b}_n .$$

THE $(x_1, \ldots, x_n)$ ARE CALLED THE COORDINATES OF $\underline{v}$ W.R.T. $\{\underline{b}_1, \ldots, \underline{b}_n\}$.

<u>PROOF:</u> SINCE THE BASIS <u>SPANS</u> GIVEN $\underline{v}$ THERE ARE $x_1, \ldots, x_n \in \mathbb{R}$ SO THAT

$$\underline{v} = x_1 \underline{b}_1 + \cdots + x_n \underline{b}_n .$$

IF $\underline{v} = y_1 \underline{b}_1 + \cdots + y_n \underline{b}_n$

THEN $\underline{0} = (x_1 - y_1)\underline{b}_1 + \cdots + (x_n - y_n)\underline{b}_n$

SO $x_1 - y_1, \ldots, x_n - y_n = 0$ BY LINEAR INDEPENDENCE.

THE ADDITION OF COORDINATES AND SCALING ARE OBVIOUS. $\square$

PROOF: BY LINEARITY

$$f(v_1\underline{v}_1 + \dots + v_n\underline{v}_n) = v_1 f(\underline{v}_1) + \dots + v_n f(\underline{v}_n)$$

$$= v_1 \cdot (a_{11}\underline{w}_1 + a_{21}\underline{w}_2 + \dots + a_{m1}\underline{w}_m)$$

$$+ v_2 \cdot (a_{12}\underline{w}_1 + a_{22}\underline{w}_2 + \dots + a_{m2}\underline{w}_m)$$

$$\vdots$$

$$+ v_n (a_{1n}\underline{w}_1 + a_{2n}\underline{w}_2 + \dots + a_{mn}\underline{w}_m)$$

$$= \boxed{(a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n)}\, \overset{w_1}{\underline{w}_1}$$

$$+ \boxed{(a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n)}\, \overset{w_2}{\underline{w}_2}$$

$$+ \vdots$$

$$+ \boxed{(a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n)}\, \overset{w_m}{\underline{w}_m}.$$

$$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}. \qquad \square$$

THEOREM: IF V HAS A BASIS CONSISTING OF n ELEMENTS, THEN ANY $n+1$ VECTORS FROM V ARE LINEARLY DEPENDENT.

<u>THEOREM</u>: IF V HAS A BASIS OF n VECTORS $\{\underline{v}_1, \ldots, \underline{v}_n\}$ THEN ANY BASIS HAS n VECTORS, SO THE DIMENSION IS WELL-DEFINED.

<u>THEOREM:</u> IF V IS SPANNED BY $\{\underline{v}_1, \dots, \underline{v}_n\}$, THEN SOME SUBSET OF $\{\underline{v}_1, \dots, \underline{v}_n\}$ IS A BASIS.

THEOREM: SUPPOSE $S \subset V$ IS A LINEARLY INDEPENDENT FINITE SET OF VECTORS. THEN EITHER THERE EXISTS A BASIS OF V CONTAINING S, OR ELSE S IS PART OF AN INFINITE LIST OF LINEARLY INDEP. VECTORS.