RECALL FROM LAST CLASS:

LET $A = \begin{pmatrix} - & \alpha_1 & - \\ & \vdots & \\ - & \alpha_n & - \end{pmatrix}_{n \times n}$; $\alpha_i \in \mathbb{R}^n$.

IF A' IS FORMED BY SWAPPING $\underline{\alpha}_i, \underline{\alpha}_j$, THEN $\det(A') = -\det(A)$.

THEOREM: IF A' IS OBTAINED FROM A BY ADDING γ TIMES $\underline{\alpha}_i$ TO $\underline{\alpha}_j$ THEN $\det(A') = \det(A)$.

THUS FAR WE'VE DESCRIBED HOW THE DETERMINANT CHANGES UNDER THE ELEMENTARY ROW OPERATIONS

(1) SWAP ROWS: $\det \times -1$.

(2) SCALE ROW BY r: $\det \times r$

(3) ADD A MULTIPLE OF ONE ROW TO ANOTHER: $\det \times 1$.

<u>THEOREM</u>: A SQUARE MATRIX A IS INVERTIBLE IFF $\det(A) \neq 0$.

IF $B \neq I_{n \times n}$ THEN THERE ARE FEWER THAN n PIVOTS SO B HAS A ROW OF ALL ZEROS $\Rightarrow \det(B) = 0$.

$\Rightarrow \det(A) = 0$.

PROOF: FIRST SUPPOSE

$\det(A), \det(B) \neq 0$.

THEN A, B ARE INVERTIBLE, SO THEIR ROW REDUCED ECHELON FORMS ARE THE IDENT.

AND THUS A AND B ARE EACH THE PRODUCT OF ELEMENTARY MATRICES.

$$\left.\begin{aligned} A &= E_1 E_2 \ldots E_n \\ B &= F_1 F_2 \ldots F_m \end{aligned}\right\} \text{ELEM. MATRICES.}$$

$AB = E_1 \ldots E_n F_1 \cdots F_m$.

IF E IS ELEM. THEN $E \cdot M$ IS THE SAME AS A ROW OPERATION ON M.

SO $\det(E \cdot M) = \det(E)\det(M)$.

SINCE WE'VE CHECKED WHAT EACH ROW OPERATION DOES TO THE DETERMINANT.

IF $\det(B) \neq 0$, THEN

B IS INVERTIBLE

$\Rightarrow A = (AB) \cdot B^{-1}$.

SO IF $\det(AB) \neq 0$ THEN

$\det(A) = \det(AB) \cdot \det(B^{-1}) \neq 0$. $\square$

THEOREM: LET $f: \mathbb{R}^m \to \mathbb{R}^n$ BE LINEAR. LET A BE THE MATRIX WITH COLUMNS

$$\begin{pmatrix} | & & | \\ f(\underline{e}_1) & \dots & f(\underline{e}_m) \\ | & & | \end{pmatrix}.$$

THEN $\forall \underline{x} \in \mathbb{R}^m$,

$$f(\underline{x}) = A \cdot \underline{x}.$$

(LINEAR MAPS $\mathbb{R}^m \to \mathbb{R}^n$ ARE GIVEN BY MULT. BY A MATRIX)

<u>EXAMPLE</u>: LET $\underline{n}$ BE A UNIT VECTOR, $|\underline{n}| = 1$.

PROJECTION ONTO DIRECTION $\underline{n}$ IS A LINEAR MAP

$$P_{\underline{n}}(\underline{x}) = (\underline{x} \cdot \underline{n}) \cdot \underline{n}.$$

$$= \underline{n} \cdot (\underline{n}^t \cdot \underline{x}) = (\underline{n} \cdot \underline{n}^t) \cdot \underline{x}$$

THEOREM: LET $f: \mathbb{R}^n \to \mathbb{R}^m$ AND $g: \mathbb{R}^m \longrightarrow \mathbb{R}^p$ BE LINEAR MAPS. LET f HAVE MATRIX A, g HAVE MATRIX B.

THEN $g \circ f$ HAS MATRIX $B \cdot A$.

DEFINITION: GIVEN $\underline{x}_1, \dots, \underline{x}_m \in \mathbb{R}^n$,

$\text{SPAN}\{\underline{x}_1, \dots, \underline{x}_m\}$

= ALL LINEAR COMBINATIONS OF $\underline{x}_1, \dots, \underline{x}_m$.

$= \{c_1 \underline{x}_1 + \dots + c_m \underline{x}_m : c_1, \dots, c_m \in \mathbb{R}\}$

PROOF:

$$\text{IMAGE}(f) = \{f(\underline{x}) : \underline{x} \in \mathbb{R}^m\}$$

$$= \{A\underline{x} : \underline{x} \in \mathbb{R}^m\}$$

$$= \left\{ \begin{pmatrix} | & & | \\ \underline{\alpha}_1 & \cdots & \alpha_m \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} : x_i \in \mathbb{R} \right\}.$$

$$= \{x_1\underline{\alpha}_1 + \cdots + x_m\underline{\alpha}_m : x_i \in \mathbb{R}\}$$

$$= \text{SPAN}(\underline{\alpha}_1, \ldots, \underline{\alpha}_m). \qquad \square$$

PROOF: f IS 1-1 MEANS

IF $f(\underline{x}) = f(\underline{y})$ THEN $\underline{x} = \underline{y}$.

BY LINEARITY, IF $f(\underline{x}) = f(\underline{y})$

THEN $f(\underline{x} - \underline{y}) = \underline{0}$, SO

f IS 1-1 IFF $f(\underline{z}) = \underline{0}$

ONLY WHEN $\underline{z} = \underline{0}$.

$$f(\underline{z}) = A \cdot \underline{z} = \begin{pmatrix} | & & | \\ \underline{\alpha}_1 & \cdots & \underline{\alpha}_m \\ | & & | \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$$

$$= z_1 \underline{\alpha}_1 + \cdots + z_m \underline{\alpha}_m .$$

THIS IS $\underline{0}$ IFF $z_1, \ldots, z_m = 0$

IS EQUIVALENT TO $\underline{\alpha}_1, \ldots, \underline{\alpha}_m$

ARE LINEARLY INDEPENDENT. $\square$

<u>PROOF</u>: SINCE f IS LINEAR

$$f(s\underline{u} + t\underline{v}) = sf(\underline{u}) + tf(\underline{v}).$$

SO $$f^{-1}(sf(\underline{u}) + tf(\underline{v}))$$

$$= s\underline{u} + t\underline{v}.$$

THIS SHOWS THAT f^{-1} IS LINEAR. $\square$

1. IF $r, s \in \mathbb{R}$, $\underline{x}$ A VECTOR

$$r\underline{x} + s\underline{x} = (r+s)\underline{x}.$$

2. $r(\underline{x}+\underline{y}) = r\underline{x} + r\underline{y}.$

3. $r \cdot (s\underline{x}) = (rs) \cdot \underline{x}$

4. $\underline{x} + \underline{y} = \underline{y} + \underline{x}$

EXAMPLES:

1. $\mathbb{R}^n$ - n TUPLES OF REAL NUMBERS
2. DEGREE $\leq n$ POLYNOMIALS.
3. $m \times n$ MATRICES
4. INFINITE SEQUENCES $a_1, a_2, a_3, \ldots$ OF REAL NUMBERS.
5. CONTINUOUS FUNCTIONS ON $[0,1]$.

<u>DEFINITION:</u> GIVEN A SET S IN A VECTOR SPACE V, THE <u>SPAN</u> OF S IS THE SET OF ALL FINITE LINEAR COMBINATIONS OF ELEMENTS OF S.

<u>EXAMPLE:</u> LET $\mathcal{P}$ BE ALL POLYNOMIALS IN x, E.G. $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$.

THIS IS A VECTOR SPACE.

THE SPACE $\mathcal{P}_n$ OF POLYS OF DEGREE $\leq n$ IS A SUBSPACE.

DEFINITION: A FUNCTION $f: V \to W$ IS LINEAR IF ITS DOMAIN AND RANGE ARE VECTOR SPACES AND $f(s\underline{x} + t\underline{y}) = sf(\underline{x}) + tf(\underline{y})$.