MAT 307: LINEAR FUNCTIONS

THE DETERMINANT IS SEPARATELY LINEAR IN EACH ROW;

IF REPLACE $\underline{\alpha}_1$ WITH $b\underline{\beta}_1 + c\underline{\gamma}_1$ THEN

$$\det\begin{pmatrix} - & b\underline{\beta}_1 + c\underline{\gamma}_1 & - \\ - & \underline{\alpha}_2 & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix} =$$

$$b \cdot \det\begin{pmatrix} - & \underline{\beta}_1 & - \\ - & \underline{\alpha}_2 & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix} + c \cdot \det\begin{pmatrix} - & \underline{\gamma}_1 & - \\ - & \underline{\alpha}_2 & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix}.$$

THIS SHOWS LINEARITY IN THE FIRST ROW; SIMILARLY ALL OTHER ROWS.

PROOF: THIS FOLLOWS FROM LINEARITY AS WE'LL SEE.

$$A' = \begin{pmatrix} - & \underline{\alpha}_1 & - \\ & \vdots & \\ - & \underline{\alpha}_i & - \\ & \vdots & \\ - & \underline{\alpha}_j + r\underline{\alpha}_i & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix}$$

$$\det(A') = \det\begin{pmatrix} - & \underline{\alpha}_1 & - \\ & \vdots & \\ - & \underline{\alpha}_j & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix}$$

$$+ \, r \det\begin{pmatrix} - & \underline{\alpha}_1 & - \\ & \vdots & \\ - & \underline{\alpha}_i & - \\ & \vdots & \\ - & \underline{\alpha}_i & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix} = 0$$

$\underline{\alpha}_i$ IN ROW j IS REPEATED.

$$= \det A.$$

THIS MEANS THAT ITS POSSIBLE TO COMPUTE THE DETERMINANT OF A MATRIX BY REDUCING IT TO A RREF, CALCULATING THE DET. OF THE REDUCED MATRIX, AND BY KEEPING TRACK OF HOW THE DETERMINANT CHANGED WITH EACH REDUCTION.

<u>PROOF</u>: LET B BE THE RREF OF A. SINCE A IS SQUARE, A IS INVERTIBLE IF AND ONLY $B = I_{n \times n} \Rightarrow \det(B) = 1$.

$\Rightarrow \det(A) \neq 0$

SINCE DOING AN ELEMENTARY ROW OPERATION MULT. BY A NON-ZERO FACTOR.

THEOREM: IF A, B ARE $n \times n$ MATRICES, THEN

$$\det(AB) = \det(A) \cdot \det(B).$$

THUS

$$\det(AB) = \det(E_1)\cdot\det(E_2)\dots\det(E_n)$$
$$\det(F_1)\dots\det(F_m)$$
$$= \det(A)\cdot\det(B).$$

IF $\det(B) = 0 \Leftrightarrow$ RREF (B) HAS A ZERO ROW

$\Leftrightarrow$ $B\underline{x} = 0$ HAS A NON-TRIVIAL SOLN

$\Rightarrow$ $AB\underline{x}$ HAS THE SAME NON-TRIVIAL SOLUTION

$\Rightarrow \det(AB) = 0.$

DEFINITION: WE SAY A MAP

$$f: \mathbb{R}^m \to \mathbb{R}^n$$

IS **LINEAR** IF, FOR ANY $a, b \in \mathbb{R}$, $\underline{x}, \underline{y} \in \mathbb{R}^m$,

$$f(a\underline{x} + b\underline{y}) = af(\underline{x}) + bf(\underline{y}).$$

PROOF:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + \dots + x_m \underline{e}_m .$$

$$f(\underline{x}) \underset{\text{LINEARITY}}{=} x_1 f(\underline{e}_1) + x_2 f(\underline{e}_2) + \dots + x_m f(\underline{e}_m)$$

$$= \begin{pmatrix} | & & | \\ f(\underline{e}_1) & \dots & f(\underline{e}_m) \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} .$$

<u>EXAMPLE:</u>

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

ROTATION BY θ IN $\mathbb{R}^2$.

PROOF:

$$(g \circ f)(\underline{x}) = g(f(\underline{x}))$$
$$= B \cdot (f(\underline{x}))$$
$$= B \cdot (A \cdot \underline{x})$$
$$= (B \cdot A) \cdot \underline{x}. \qquad \square$$

THEOREM: LET $f: \mathbb{R}^m \to \mathbb{R}^n$ HAVE MATRIX A. THEN THE IMAGE OF f IS THE SPAN OF THE COLUMNS OF A.

<u>THEOREM:</u> $f: \mathbb{R}^m \to \mathbb{R}^n$

LINEAR IS 1-1 IFF THE COLUMNS OF THE MATRIX A ARE LINEARLY INDEPENDENT.

<u>THEOREM</u>: IF $f: \mathbb{R}^n \to \mathbb{R}^n$ IS 1-1 THEN f IS INVERTIBLE AND LINEAR ON ITS RANGE.

VECTOR SPACES:

AN ABSTRACT VECTOR SPACE $V/\mathbb{R}$ IS A SET WITH ADDITION AND SCALAR MULT SATISFYING THE FOLLOWING PROPERTIES.

5. $(\underline{x}+\underline{y})+\underline{z} = \underline{x}+(\underline{y}+\underline{z})$.

6. THERE IS A $\underline{0}$ VECTOR,

$$\underline{x}+\underline{0} = \underline{0}+\underline{x} = \underline{x}$$

7. MULTIPLICATION BY 1:

$$1 \cdot \underline{x} = \underline{x}.$$

8. $(-1)\cdot\underline{x} = -\underline{x}$ AND $\underline{x}+(-\underline{x}) = \underline{0}$.

9. $0 \cdot \underline{x} = \underline{0}$.

<u>DEFINITION</u>: A SUBSET $W \subset V$ OF A VECTOR SPACE IS A SET CLOSED UNDER ADDITION AND SCALAR MULT.

E.G. A LINE THROUGH 0 IN $\mathbb{R}^2$ IS A SUBSPACE.

THEOREM: THE SPAN OF A SET $\underline{S}$ IS A SUBSPACE

PROOF: IF $\underline{x}, \underline{y}$ ARE TWO ELEMENTS OF SPAN($\underline{S}$).

THEN $\underline{x} = a_1 \underline{s}_1 + \dots + a_n \underline{s}_n$

$\underline{y} = b_1 \underline{t}_1 + \dots + b_m \underline{t}_m$

$$\alpha x + \beta y = \alpha a_1 \underline{s}_1 + \dots + \alpha a_n \underline{s}_n + \beta b_1 \underline{t}_1 + \dots + \beta b_m \underline{t}_m.$$

IS STILL A LINEAR COMBINATION. $\square$

EXAMPLE: LET V BE CONTINUOUS FNS ON $[0,1]$.

LET $W = \left\{ f \in V : \int_0^1 f \, dx = 0 \right\}$.

THEN W IS A SUBSPACE.