<u>DEFINITION</u>: LET A, B BE SQUARE MATRICES OF THE SAME DIMENSION SUCH THAT $AB = B \cdot A = I$. THEN $B = A^{-1}$.

PROOF: IF $\underline{x} = A^{-1} \cdot \underline{b}$

THEN $A \cdot \underline{x} = A \cdot (A^{-1} \cdot \underline{b})$

$= (A \cdot A^{-1}) \cdot \underline{b} = \underline{b}.$

IF $\underline{x}'$ IS ANY SOLUTION TO $A\underline{x}' = \underline{b}$

THEN $\underbrace{A^{-1} A}_{I} \underline{x}' = A^{-1} \cdot \underline{b}$

$\underline{I} \cdot \underline{x}' = \underline{x}'.$ $\square$

MATRIX INVERSION:

LET A BE A $n \times n$ INVERTIBLE MATRIX. STARTING FROM $(A \mid I_{n \times n})$ PERFORM ELEMENTARY ROW OPERATIONS ON A UNTIL $I_{n \times n}$ IS REACHED.

THIS LEAVES

$$(I_{n \times n} \mid A^{-1}).$$

<u>THEOREM:</u> IF A IS A SQUARE MATRIX, $A\underline{x} = \underline{0}$ HAS THE UNIQUE SOLUTION $\underline{x} = 0$ IFF A IS INVERTIBLE.

NOTE: IF

$$A = \begin{pmatrix} | & & | \\ \underline{\alpha}_1 & \dots & \underline{\alpha}_n \\ | & & | \end{pmatrix}$$ AND

$c_1 \underline{\alpha}_1 + \dots + c_n \underline{\alpha}_n = \underline{0}$ THEN

$$A \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1 \underline{\alpha}_1 + \dots + c_n \underline{\alpha}_n = \underline{0},$$

PROOF: A INVERTIBLE

$\Leftrightarrow$ $A\underline{x}=\underline{0}$ HAS $\underline{x}=0$ AS A UNIQUE SOLUTION.

$\Leftrightarrow$ $B=\mathrm{rref}(A)$ HAS n PIVOTS, NO 0 ROWS.

$\Leftrightarrow$ $B=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ & 1 & \ddots & 0 \\ 0 & & \ddots & 1 \end{pmatrix} = I_{n\times n}$.

EACH ROW OPERATION CAN BE REPRESENTED AS A MULTIPLICATION ON THE LEFT BY AN ELEMENTARY MATRIX

SO $E_m \cdot E_{m-1} \cdot \ldots \cdot E_1 A = I_{n\times n}$.

WHERE $E_1 \cdot A$ IS THE MATRIX AFTER FIRST ROW OP, $E_2 \cdot E_1 \cdot A$, 2 OPERATIONS, ETC.

ON THE RIGHT WE HAVE $E_m \cdot E_{m-1} \cdots E_1 \cdot I_{n\times n}$.
$= B$.

$(A, I_{n\times n})$

SO $(I_{n\times n}, B)$, THE FINAL OUTCOME

SATISFIES

$$B \cdot A = I_{n\times n}.$$

TO CONCLUDE $B=A^{-1}$, WE MUST HAVE $AB=I_{n\times n}$, ALSO.

$$B = E_m E_{m-1} \cdots E_1.$$

EACH ELEMENTARY ROW OPERATION CAN BE UNDONE,

SAY $E_i^{-1} \cdot E_i = I_{n\times n}$

E_i^{-1} IS ALSO ELEMENTARY.

FIRST UNDO E_m, THEN E_{m-1}, ...

$$B = E_m \cdot E_{m-1} \cdot \ldots \cdot E_1$$

$E_1^{-1} E_2^{-1} \ldots E_m^{-1} \cdot B$

$= \cancel{E_1^{-1}} \cancel{E_2^{-1}} \ldots ((\underbrace{\cancel{E_m^{-1} \cdot E_m}}_{I}) \cancel{E_{m-1}}) \ldots \cancel{E_1}$

$= I_{n\times n}$.

SAY $C = E_1^{-1} E_2^{-1} \ldots E_m^{-1}$.

SO $C \cdot B = I$.

SIMILARLY $B \cdot C = \cancel{E_m} E_{m-1} \cdots \cancel{E_1} \cdot \cancel{E_1^{-1}} \cdots \cancel{E_m^{-1}}$

$= I_{n\times n}$.

$A \cdot (BC) = A \cdot I_{n\times n}$.

$= \overset{I}{(AB)} \cdot C$

SO $A = C$. THUS $AB = B \cdot A = I_{n\times n}$. $\square$

E.G.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\det A = 1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 1 \cdot 5 \cdot 9 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7$$

LET A <u>PERMUTATION</u> OF $\{1, 2, \ldots, n\}$ BE A MAP

$$f : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

SUCH THAT EACH VALUE IS TAKEN ONCE.

$f(i)$ IS THE ENTRY TAKEN IN ROW i.

GIVEN $A_{n \times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\boxed{\det A = \sum_{\substack{f \text{ PERM} \\ \text{OF } \{1,2,\ldots,n\}}} \text{sign}(f) \cdot a_{1f(1)} \cdots a_{nf(n)}.}$$

IT IS POSSIBLE TO PROVE THAT THE ROW EXPANSION AND PERMUTATION FORMULAE ARE EQUIVALENT. (EXERCISE).

PROOF: LET

$$A = \begin{pmatrix} - \underline{\alpha}_1 - \\ \vdots \\ - \underline{\alpha}_n - \end{pmatrix}.$$

EXPAND BY ROW i.

$$\det \begin{pmatrix} - \underline{\alpha}_1 - \\ \vdots \\ - a\underline{\beta}_i + b\underline{\gamma}_i - \\ \vdots \\ - \underline{\alpha}_n - \end{pmatrix} = \sum_{j=1}^{n} (-1)^{i+j} (a\beta_{ij} + b\gamma_{ij}) \det A_{ij}.$$

$$= \sum_{j=1}^{n} (-1)^{i+j} (a\beta_i \det A_{ij} + b\gamma_i \det A_{ij})$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a\beta_i \det A_{ij} + \sum_{j=1}^{n} (-1)^{i+j} b\gamma_i \det A_{ij}.$$

$$a \cdot \det \begin{pmatrix} \underline{\alpha}_1 \\ \vdots \\ \underline{\alpha}_{i-1} \\ \underline{\beta}_i \\ \vdots \\ \underline{\alpha}_n \end{pmatrix} + b \det \begin{pmatrix} \underline{\alpha}_1 \\ \vdots \\ \underline{\gamma}_i \\ \vdots \\ \underline{\alpha}_n \end{pmatrix}.$$

THE SAME FORMULA HOLDS FOR COLUMN EXPANSION, SEE THE TEXT.

PROOF IS BY INDUCTION ON THE NUMBER OF ROWS;

<u>2x2 CASE:</u> $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$ ✓

$\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad.$

LET THE ROWS BE i, j,

AND LET k BE ANY OTHER

ROW. EXPANDING BY

ROW k OBTAINS

$$\det A = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} \det A_{jk}$$

A'_{jk} HAS TWO ROWS SWAPPED FROM A_{jk}

$$= - \sum_{j=1}^{n} (-1)^{j+k} a_{jk} \det A'_{jk}.$$

BY INDUCTION

$$= - \det A'. \quad \square$$